UNIT 3

MULTIVARIABLE CALCULUS (DIFFERENTIATION)

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Evaluate $\lim_{(x,y)\to(1,2)} \frac{x^2y}{x+y^2}$.

Answer : June/July-17, Q7

Given limit is,

$$\lim_{(x,\,y)\to(1,2)} \frac{x^2y}{x+y^2}$$

$$\Rightarrow \lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2y}{x+y^2} = \lim_{x\to 1}\left[\lim_{y\to 2} \frac{x^2y}{x+y^2}\right]$$

$$= \lim_{x\to 1}\left[\frac{x^2(2)}{x+(2)^2}\right]$$

$$= \lim_{x\to 1}\left[\frac{2x^2}{x+4}\right]$$

$$= \frac{2(1)^2}{1+4} = \frac{2}{5}$$

$$\therefore \lim_{(x,\,y)\to(1,2)} \frac{x^2y}{x+y^2} = \frac{2}{5}$$

Q2. Show that $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ doesn't exist.

Answer : Dec.-16, Q7

Given that,

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Let,

$$p_1 = \lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{xy^3}{x^2+y^6}\right]$$

$$= \lim_{y\to 0}\left[\lim_{x\to 0} \frac{xy^3}{x^2+y^6}\right]$$

$$= \lim_{y\to 0}\left[\frac{0.y^3}{0+y^6}\right]$$

$$= 0$$

$$\therefore \quad p_1 = 0$$

Let,

$$p_2 = \lim_{\substack{y \to 0 \\ x \to 0}} \left[\frac{xy^3}{x^2 + y^6} \right]$$

$$= \lim_{x \to 0} \left[lt_{y \to 0} \frac{xy^3}{x^2 + y^6} \right]$$

$$= \lim_{x \to 0} \left[\frac{x(0)}{x^2 + 0} \right]$$

$$= 0$$

$\therefore \quad p_2 = 0$

Since, $p_1 = p_2 = 0$, the limit doesn't exists.

Q3. Determine $\underset{(x,y)\to(0,0)}{\text{Lt}} \frac{xy^2}{x^2+y^4}$ if it exists.

Answer : **Dec.-15, Q5**

Given limit is,

$$\underset{(x,y)\to(0,0)}{Lim} \frac{xy^2}{x^2 + y^4}$$

(i) $$P_1 = \underset{\substack{x\to0\\y\to0}}{Lim} \left[\frac{xy^2}{x^2 + y^4} \right]$$

$$= \underset{x\to0}{Lim} \left[\underset{y\to0}{Lim} \frac{xy^2}{x^2 + y^4} \right]$$

$$= \underset{x\to0}{Lim} \left[\frac{0}{x^2 + 0} \right]$$

$$= 0$$

$\therefore P_1 = 0$

(ii) $$P_2 = \underset{\substack{x\to0\\y\to0}}{Lim} \frac{xy^2}{x^2 + y^4}$$

$$= \underset{y\to0}{Lim} \left[\underset{x\to0}{Lim} \frac{xy^2}{x^2 + y^4} \right]$$

$$= \underset{y\to0}{Lim} \left[\frac{0}{0 + y^4} \right]$$

$$= 0$$

(iii) Let, $y = mx$

$$P_3 = \underset{\substack{x\to0\\y\to mx}}{Lim} \frac{xy^2}{x^2 + y^4} = \underset{x\to0}{Lim} \frac{xm^2x^2}{x^2 + m^4x^4} = \underset{x\to0}{Lim} \frac{m^2x}{1 + m^4x^2} = 0$$

(iv) Let, $y = mx^2$

$$P_4 = \underset{\substack{x\to0\\y\to mx^2}}{Lim} \frac{xy^2}{x^2 + y^4} = \underset{x\to0}{Lim} \frac{xm^2x^4}{x^2 + m^4x^3} = \underset{x\to0}{Lim} \frac{m^2x^3}{1 + m^4x^5} = 0$$

$P_1 = P_2 = P_3 = P_4$

$\therefore \quad \underset{\substack{x\to0\\y\to0}}{Lim} \frac{xy^2}{x^2 + y^4} = 0$

Q4. Determine $\underset{(x,y)\to(1,-1)}{\text{Lim}} x^2 - y^2$.

Answer : **April-16, Q5**

Given limit is,

$$\underset{(x,y)\to(1,-1)}{Lim} x^2 - y^2$$

$$\Rightarrow \quad \underset{(x,y)\to(1,-1)}{Lim} = \underset{\substack{x\to1\\y\to-1}}{Lim} x^2 - y^2$$

$$= \underset{x\to1}{Lim} \left[\underset{y\to-1}{Lim} x^2 - y^2 \right]$$

$$= \underset{x\to1}{Lim} \left[x^2 - (-1)^2 \right]$$

$$= \underset{x\to1}{Lim} x^2 - 1$$

$$= (1)^2 - 1$$

$$= 0$$

$\therefore \quad \underset{(x,y)\to(1,-1)}{Lim} = 0$

Q5. Show that $f(x, y) = \begin{cases} \frac{x-y}{x+y}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$ is discontinuous at the point (0, 0).

Answer : **Dec.-12, Q7**

The given function is,

$$f(x, y) = \begin{cases} \frac{x-y}{x+y}; & (x, y) \neq (0,0) \\ 0; & (x, y) = (0,0) \end{cases}$$

To check the continuity of the function $f(x, y)$, directly choose the path $y \to mx$.

$$\underset{\substack{y\to mx\\x\to0}}{Lt} f(x, y) = \underset{\substack{y\to mx\\x\to0}}{Lt} \left[\frac{x-y}{x+y} \right]$$

$$= \underset{x\to0}{Lt} \left[\underset{y\to mx}{Lt} \frac{x-y}{x+y} \right]$$

$$= \underset{x\to0}{Lt} \left[\frac{x-mx}{x+mx} \right]$$

$$= \underset{x\to0}{Lt} \left[\frac{x(1-m)}{x(1+m)} \right]$$

$$= \underset{x\to0}{Lt} \frac{1-m}{1+m}$$

i.e., the function depends on 'm' values only.

$\therefore$ The limit $x \to 0$ does not exists.

Hence, $f(x, y)$ is not continuous at (0, 0).

Q6. If x = r cos θ and y = r sin θ then find $\frac{\partial r}{\partial x}$.

Answer :

Given that,

$$x = r\cos\theta \quad \ldots (1)$$

$$y = r\sin\theta \quad \ldots (2)$$

Adding equations (1) and (2),

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2$$

$$x^2 + y^2 = r^2 \quad \ldots (3)$$

Differentiating equation (3) partially with respect to 'x',

$$\Rightarrow \quad 2x = 2r\frac{\partial r}{\partial x}$$

$$\Rightarrow \quad x = r\frac{\partial r}{\partial x}$$

$$\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Q7. If $u = \frac{y}{z} + \frac{z}{x}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Answer :

Given equation is,

$$u = \frac{y}{z} + \frac{z}{x} \quad \ldots (1)$$

In equation (1), u is dependent variable and x, y, z are independent variables.

(i) To Calculate $\frac{\partial u}{\partial x}$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{z} + \frac{z}{x}\right) = \left(0 + z\left(\frac{-1}{x^2}\right)\right)$$

[$\because$ y, z are constants]

$$= \frac{-z}{x^2} \quad \ldots (2)$$

(ii) To Calculate $\frac{\partial u}{\partial y}$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{z} + \frac{z}{x}\right)$$

$$= \left(\frac{1}{z} + 0\right) \qquad [\because x, z \text{ are constants}]$$

$$= \frac{1}{z} \quad \ldots (3)$$

(iii) To Calculate $\frac{\partial u}{\partial z}$

$$\Rightarrow \quad \frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{z} + \frac{z}{x}\right) = \left(y\left(\frac{-1}{z^2}\right) + \frac{1}{x}\right)$$

[$\because$ x, y are constants]

$$= \frac{-y}{z^2} + \frac{1}{x} \quad \ldots (4)$$

Consider,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(\frac{-z}{x^2}\right) + y\left(\frac{1}{z}\right) + z\left(\frac{-y}{z^2} + \frac{1}{x}\right)$$

$$= \frac{-z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$$

$$= 0$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Q8. If $u = \frac{x+y}{xy}$ find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Answer :

Given function is,

$$u = \frac{x+y}{xy}$$

$$\Rightarrow \quad u = \frac{x}{xy} + \frac{y}{xy}$$

$$\Rightarrow \quad u = \frac{1}{y} + \frac{1}{x} \quad \ldots (1)$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial u}{\partial x} = -\frac{1}{x^2}$$

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial u}{\partial y} = -\frac{1}{y^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{1}{x^2} \text{ and } \frac{\partial u}{\partial y} = -\frac{1}{y^2}.$$

Q9. If $x^y + y^x = 1$, then find $\frac{dy}{dx}$.

Answer :

Given function is,

$$x^y + y^x = 1$$

Let, $f = x^y + y^x - 1 = 0 \quad \ldots (1)$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y - 0 \qquad \left[\because \frac{d}{dx} a^x = a^x \log a\right]$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiating equation (1), partially with respect to 'y',

$$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1} - 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$\frac{dy}{dx}$ is given as,

$$\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

Substituting the corresponding values in above equation,

$$\frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$$

$$\therefore \quad \frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$$

Q10. Find $\frac{du}{dx}$ if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$.

Answer :

Given that,

$$u = \sin(x^2 + y^2) \qquad \ldots (1)$$

$$a^2x^2 + b^2y^2 = c^2 \qquad \ldots (2)$$

Partially differentiating equation (1) with respect to 'x',

$$\frac{\partial u}{\partial x} = 2x\cos\left(x^2 + y^2\right)$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x\cos\left(x^2 + y^2\right)$$

Partially differentiating equation (1) with respect to 'y',

$$\frac{\partial u}{\partial y} = 2y\cos\left(x^2 + y^2\right)$$

$$\therefore \quad \frac{\partial u}{\partial y} = 2y\cos\left(x^2 + y^2\right)$$

Differentiating equation (2) with respect to 'x',

$$2a^2x + 2b^2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{2b^2 y dy}{dx} = -2a^2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-a^2x}{b^2y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \qquad [\because \text{ From chain rule}] \qquad \ldots (3)$$

Substituting the corresponding values in equation (3),

$$\frac{du}{dx} = 2x\cos\left(x^2 + y^2\right) + \left[2y\cos\left(x^2 + y^2\right)\left[\frac{-a^2x}{b^2y}\right]\right]$$

$$= 2x\cos\left(x^2 + y^2\right) + \frac{\left(-2xa^2\cos\left(x^2 + y^2\right)\right)}{b^2}$$

$$= 2x\cos\left(x^2 + y^2\right)\left[\frac{b^2 - a^2}{b^2}\right]$$

$$\therefore \quad \frac{du}{dx} = 2x\cos\left(x^2 + y^2\right)\left[\frac{b^2 - a^2}{b^2}\right]$$

Q11. If $u = x^3ye^z$, where $x = t$, $y = t^2$ and $z = \log_e t$, find $\frac{du}{dt}$ at $t = 2$.

Answer :

Given that,

$$u = x^3ye^z$$

$$x = t,\ y = t^2 \text{ and } z = \log_e t$$

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2t \text{ and } \frac{dz}{dt} = \frac{1}{t}$$

Differentiating 'u' partially with respect to x, y and z,

$$\frac{\partial u}{\partial x} = 3x^2ye^z \ ; \quad \frac{\partial u}{\partial y} = x^3e^z \ ; \quad \frac{\partial u}{\partial z} = x^3ye^z$$

From the definition of total derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 3x^2ye^z.1 + x^3e^z.2t + x^3ye^z.\frac{1}{t}$$

$$\frac{du}{dt} = 3x^2ye^z + x^3e^z 2t + \frac{x^3y.e^z}{t}$$

Substituting x, y and z values in above equation,

$$\frac{du}{dt} = 3t^2.t^2e^{\log_e t} + t^3e^{\log_e t}.2t + \frac{t^3.t^2.e^{\log_e t}}{t}$$

$$= 3t^4e^{\log_e t} + 2t^4e^{\log_e t} + t^4e^{\log_e t}$$

$$= 3t^4.t + 2t^4.t + t^4.t \qquad [\because \ e^{\log_e t} = t\ ]$$

$$= 3t^5 + 2t^5 + t^5$$

$$\therefore \quad \frac{du}{dt} = 6t^5$$

For $t = 2$,

$$\therefore \quad \frac{du}{dt} = 6(2)^5 = 192.$$

Verification by Direct Substitution

$$u = x^3ye^z$$

$$= t^3t^2\,e^{\log_e t}$$

$$u = t^5\,e^{\log_e t}$$

Differentiating above equation with respect to 't',

$$\frac{du}{dt} = t^5 + e^{\log_e t}\, 5t^4$$
$$= t^5 + 5t^5 = 6t^5$$

For $t = 2$,

$$\therefore \quad \frac{du}{dt} = 6(2)^5 = 192.$$

Q12. Find du/dt when $u = x^2 y$, $x = t^2$ and $y = e^t$.

Answer :

Given functions are,

$$u = x^2 y \qquad \ldots (1)$$
$$x = t^2 \qquad \ldots (2)$$
$$y = e^t \qquad \ldots (3)$$

Differentiating equation (1) partially with respect to x and y,

$$\frac{\partial u}{\partial x} = 2xy, \quad \frac{\partial u}{\partial y} = x^2$$

Differentiating equation (2) partially with respect to t,

$$\frac{\partial x}{\partial t} = 2t$$

Differentiating equation (3) partially with respect to t,

$$\frac{\partial y}{\partial t} = e^t$$

From chain rule,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \qquad \ldots (4)$$

Substituting the corresponding values in equation (4),

$$\frac{du}{dt} = (2xy)\,(2t) + (x^2)\,(e^t)$$
$$\Rightarrow \quad \frac{du}{dt} = 4xyt + x^2 e^t \qquad \ldots (5)$$

Substituting equations (2) and (3) in equation (5),

$$\frac{du}{dt} = 4(t^2)\,(e^t)t + (t^2)^2 e^t$$
$$= 4t^3 e^t + t^4 e^t$$
$$= t^3 e^t\,(4 + t)$$
$$\therefore \quad \frac{du}{dt} = t^3 e^t\,(4 + t).$$

Q13. Find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ if $u = f(x + y, x - y)$.

Answer :

Given that,

$$u = f(x + y, x - y)$$

Let, $a = x + y$ and

$$b = x - y$$

Then, $u = f(a, b)$

$$\frac{\partial a}{\partial x} = 1, \quad \frac{\partial a}{\partial y} = 1$$

$$\frac{\partial b}{\partial x} = 1, \quad \frac{\partial b}{\partial y} = -1$$

Applying chain rule of partial differentiation,

$$\text{i.e.,} \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial x}$$
$$= \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(1)$$
$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \qquad \ldots (1)$$

Similary,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial y} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial y}$$
$$= \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(-1)$$
$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial b} \qquad \ldots (2)$$

Adding equations (1) and (2),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} + \frac{\partial u}{\partial a} - \frac{\partial u}{\partial b}$$
$$= 2\frac{\partial u}{\partial a}$$
$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial a}$$

Q14. Find $\frac{du}{dt}$ when $u = x^2 + y^2$, $x = at^2$, $y = 2at$.

Answer :

Given functions are,

$$u = x^2 + y^2 \qquad \ldots (1)$$
$$x = at^2 \qquad \ldots (2)$$
$$y = 2at \qquad \ldots (3)$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial u}{\partial x} = 2x$$

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial u}{\partial y} = 2y$$

Differentiating equation (2) partially with respect to 't',

$$\frac{\partial x}{\partial t} = a(2t) = 2at$$

Differentiating equation (3) partially with respect to 't',

$$\frac{\partial y}{\partial t} = 2a$$

From chain rule,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \qquad \ldots (4)$$

Substituting the corresponding values in equation (4),

$$= 2x(2at) + 2y(2a)$$

$$= 4xat + 4ay$$

$$\therefore \quad \frac{du}{dt} = 4xat + 4ay \qquad \ldots (5)$$

Substituting equations (2) and (3) in equation (5),

$$\frac{du}{dt} = 4(at^2)at + 4a(2at)$$

$$= 4a^2t^3 + 8a^2t$$

$$= 4at\,(t^2 + 2)$$

$$\therefore \quad \frac{du}{dt} = 4at\,(t^2 + 2).$$

Q15. If u = f(y – z, z – x, x – y), find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

Answer :

Given that,

$$u = f(y - z, z - x, x - y)$$

Let,

$$a = y - z$$
$$b = z - x$$
$$c = x - y$$

Then,

$$u = f(a, b, c)$$
$$a = y - z;\ b = z - x;\ c = x - y$$

$$\frac{\partial a}{\partial x} = 0; \quad \frac{\partial b}{\partial x} = -1; \quad \frac{\partial c}{\partial x} = 1$$

$$\frac{\partial a}{\partial y} = 1; \quad \frac{\partial b}{\partial y} = 0; \quad \frac{\partial c}{\partial y} = -1$$

$$\frac{\partial a}{\partial z} = -1; \quad \frac{\partial b}{\partial z} = 1; \quad \frac{\partial c}{\partial z} = 0$$

Applying chain rule of partial differentiation.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x}$$

$$= \frac{\partial u}{\partial a}(0) + \frac{\partial u}{\partial b}(-1) + \frac{\partial u}{\partial c}(1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial c} - \frac{\partial u}{\partial b} \qquad \ldots (1)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial y}$$

$$= \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(0) + \frac{\partial u}{\partial c}(-1)$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} \qquad \ldots (2)$$

And $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z}$$

$$= \frac{\partial u}{\partial a}(-1) + \frac{\partial u}{\partial b}(1) + \frac{\partial u}{\partial c}(0)$$

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial b} - \frac{\partial u}{\partial a} \qquad \ldots (3)$$

Adding equations (1), (2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial c} - \frac{\partial u}{\partial b} + \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} - \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Q16. Define the Jacobian of the functions $u_1, u_2, \ldots, u_n$ of n variables $x_1, x_2, \ldots, x_n$.

Answer :

Definition

If $u_1, u_2, \ldots, u_n$ be the functions of 'n' variables $x_1, x_2, \ldots, x_n$, then the determinant,

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ and is denoted by $J(u_1, u_2, \ldots, u_n)$ or $\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)}$

Q17. If x = uv, y = $\frac{u}{v}$ then find $\frac{\partial(x, y)}{\partial(u, v)}$

Answer :

Given functions are,

$$x = uv \qquad \ldots (1)$$

$$y = \frac{u}{v} \qquad \ldots (2)$$

Differentiating the equation (1) with respect to u, v,

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u$$

Differentiating equation (2) with respect to u, v,

$$\frac{\partial y}{\partial u} = \frac{1}{v}, \quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Substituting the corresponding values in above equation,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$= v\left(\frac{-u}{v^2}\right) - u\left(\frac{1}{v}\right)$$

$$= -\frac{uv}{v^2} - \frac{u}{v}$$

$$= \frac{-u}{v} - \frac{u}{v}$$

$$= \frac{-2u}{v}$$

$$\therefore \frac{\partial(x,y)}{\partial(u.v)} = \frac{-2u}{v}$$

Q18. If x = u(1 + v) and y = v(1 + u), find ∂(x, y) / ∂(u, v).

Answer :

Given functions are,

$$x = u(1+v) \quad \ldots (1)$$

$$y = v(1+u) \quad \ldots (2)$$

Differentiating equation (1) partially with respect to u and v,

$$\frac{\partial x}{\partial u} = 1 + v, \quad \frac{\partial x}{\partial v} = u$$

Differentiating equation (2) partially with respect to u and v,

$$\frac{\partial y}{\partial u} = v, \quad \frac{\partial y}{\partial v} = 1 + u$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \ldots (3)$$

Substituting the corresponding values in equation (3),

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + v + u + uv - uv$$

$$= 1 + u + v$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = 1 + u + v.$$

Q19. If x = r cosθ, y = r sinθ, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

OR

Find the Jacobian of the transformation x = r cosθ, y = r sinθ.

OR

Find $\frac{\partial(x,y)}{\partial(r,\theta)}$, if x = r cos θ, y = r sin θ.

Answer :

Given that,

$$x = r\cos\theta \quad \ldots (1)$$

$$y = r\sin\theta \quad \ldots (2)$$

Differentiating equations (1) and (2) partially with respect to 'r'.

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

Differentiating equations (1) and (2) partially with respect to 'θ',

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad ; \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \cos\theta\,(r\cos\theta) - (-r\sin\theta)\sin\theta$$

$$= r(\sin^2\theta + \cos^2\theta)$$

$$= r$$

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

Q20. If x = r cos θ, y = r sinθ, find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

OR

If x = r cos θ, y = r sinθ, then find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Answer :

Given that,

$$x = r\cos\theta \quad \ldots (1)$$

$$y = r\sin\theta \quad \ldots (2)$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}}$$

[$\because$ From the properties of Jacobians]

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}} \quad \ldots (3)$$

Differentiating equations (1) and (2) partially with respect to 'r'

$$\frac{\partial x}{\partial r} = \cos\theta \; ; \; \frac{\partial y}{\partial r} = \sin\theta$$

Differentiating equations (1) and (2) partially with respect to 'θ'

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \; ; \; \frac{\partial y}{\partial \theta} = r\cos\theta$$

Substituting the corresponding values in equation (3),

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}}$$

$$= \frac{1}{r\cos\theta(\cos\theta) - \sin\theta(-r\sin\theta)}$$

$$= \frac{1}{r\cos^2\theta + r\sin^2\theta}$$

$$= \frac{1}{r(\cos^2\theta + \sin^2\theta)}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}.$$

Q21. If x = u(1 – v), y = uv, show that JJ' = 1.

Answer :

Given that,

$$x = u(1-v), y = uv$$

$$\Rightarrow \quad x = u - uv = u - y$$

$$\Rightarrow \quad u = x + y$$

$$\Rightarrow \quad y = (x+y)v$$

$$\Rightarrow \quad v = \frac{y}{x+y}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = u - uv \Rightarrow \frac{\partial x}{\partial u} = 1 - v, \; \frac{\partial x}{\partial v} = -u$$

$$y = uv \Rightarrow \frac{\partial y}{\partial u} = v, \; \frac{\partial y}{\partial v} = u$$

$$\therefore \quad J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$u = x + y$$

$$y = uv \Rightarrow v = \frac{y}{u} = \frac{y}{x+y}$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y)-y}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} - \frac{-y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore \quad JJ' = u.\frac{1}{u} = 1.$$

Q22. Write the expressions for Taylor's series for function of two variables.

Answer :

The expression for Taylor's series for function of two variables is,

$$f(x,y) = f(a,b) + \left[\frac{(x-a)}{1!} f_x(a,b) + \frac{(y-b)}{1!} f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right] + ...$$

Q23. Find the Taylor series expansion of x^y near the point (1, 1) upto the first degree terms.

Answer :

Given function is,

$f(x, y) = x^y$

$f(1, 1) = 1^{(1)} = 1$

$f_x(x, y) = yx^{y-1}$; $f_x(1, 1) = 1(1)^{1-1} = 1$

$f_y(x, y) = x^y \log x$; $f_y(1, 1) = 0$

From Taylor's series,

$$\Rightarrow f(x, y) = f(1, 1) + (x - 1) f_x(1, 1) + (y - 1) f_y(1, 1)$$

$$x^y = 1 + (x - 1)(1) + (y - 1)0$$

$$= 1 + x - 1 + 0$$

$$\therefore \quad x^y = x.$$

Q24. Define maximum and minimum values for a function of two variables.

Answer :

Maximum Value

A function $f(x, y)$ is said to have maximum value at $x = a$ and $y = b$ if and only if,

$f(a, b) > f(a + h, b + k)$

Where,

h, k are small values.

Minimum Value

A function $f(x, y)$ is said to have minimum value at $x = a$ and $y = b$ if and only if,

$f(a, b) < f(a + h, b + k)$

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

3.1 FUNCTIONS OF TWO VARIABLES, LIMITS AND CONTINUITY

Q25. Explain function of several variables.

Answer :

Function of Several Variables

A function which contains more than one variable is called function of several variables.

(i) **Function of Two Variables:** A function which contains two variables is called function of two variables.

It is of the form,

$z = f(x,y)$

Example: $z = x^2 + y^2$

(ii) **Function of Three Variables:** A function which has three variables is called function of three variables.

It is of the form,

$v = f(x, y, z)$

Example: $v = x^2 + y^2 + z^2$.

Q26. Define limit and continuity of a function. Also mention the steps involved in obtaining the limit of a function of two variables.

Answer :

Limit

A function $f(x, y)$ is said to have limit L as 'x' tends to 'a' and 'y' tends to 'b' i.e,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If for every positive number $\varepsilon > 0$, There exists a corresponding number $\delta > 0$, such that,

$|f(x, y) - L| < \varepsilon$ when $|x - a| < \delta$ and $|y - b| < \delta$

Continuity

A function $f(x, y)$ at a point (a, b) is said to be continuous, if it is continuous at each point such that,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a, b)$$

Procedure

Step 1

Initially, determine the lim $f(x, y)$ along path p_1

i.e, $x \to a$ and $x \to b$

Step 2

In the next step, determine lm $f(x, y)$ along path p_2

i.e, $y \to b$ and $x \to a$

Step 3

In this step obtain the limit along path $y = mx$ or $y = mx^n$, if the value of a and b are 0.

Q27. If $f(x, y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$, show that $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ at (0, 0).

Answer : **June/July-17, Q14(a)**

Given that,

$$f(x, y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

$$f(x, y) = \frac{x^2 y(x-y)}{x^2+y^2}$$

$$f(0, 0) = 0 \quad \ldots (1)$$

$$\Rightarrow \quad f(\Delta x, 0) = \frac{(\Delta x)^2(0)(\Delta x - 0)}{(\Delta x)^2 + (0)^2}$$

$$= \frac{(0)}{(\Delta x)^2}$$

$$f(\Delta x, 0) = f(x, 0) = 0 \quad \ldots (2)$$

$$\Rightarrow \quad f(0, \Delta y) = \frac{(0)^2 \Delta y(0 - \Delta y)}{(0)^2 (\Delta y)^2}$$

$$= \frac{0}{(\Delta y)^2}$$

$$\therefore \quad f(0, \Delta y) = f(0, y) = 0 \quad \ldots (3)$$

Since,

$$\frac{\partial f}{\partial x} = f_x(x, y)$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \frac{0 - 0}{\Delta x} \quad \text{[From equation (1) and (2)]}$$

$$= 0$$

$$\therefore f_x(0, 0) = 0 \quad \ldots (4)$$

$$f_x(0, y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{(\Delta x)^2 y(\Delta x - y)}{(\Delta x)^2 + y^2} - 0}{\Delta x}$$

[$\because$ From equation (3)]

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 y(\Delta x - y)}{\Delta x((\Delta x)^2 + y^2)}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x) y(\Delta x - y)}{(\Delta x)^2 + y^2}$$

$$= \frac{(0) y(0 - y)}{(0)^2 + y^2} = \frac{0(0 - y)}{y^2}$$

$$= \frac{0}{y^2} = 0$$

$$\Rightarrow \quad f_x(0, \Delta y) = 0$$

$$\therefore f_x(0, \Delta y) = 0 \quad \ldots (5)$$

Similarly,

$$\frac{\partial f}{\partial y} = f_y(x, y)$$

$$= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y}$$

[From equations (1) and (2)]

$$= 0$$

$$\therefore f_y(0, 0) = 0 \quad \ldots (6)$$

$$f_y(x, 0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\dfrac{x^3 (\Delta y)(x - \Delta y)}{x^2 + (\Delta y)^2} - 0}{\Delta y}$$

[From equation (3)]

$$= \lim_{\Delta y \to 0} \frac{x^3 (\Delta y)(x - \Delta y)}{\Delta y(x^2 + (\Delta y)^2)}$$

$$= \lim_{\Delta y \to 0} \frac{x^3 (x - \Delta y)}{x^2 + (\Delta y)^2}$$

$$= \frac{x^3 (x - 0)}{x^2 + (0)^2}$$

$$= \frac{x^4}{x^2} = x^2$$

$$\Rightarrow \quad f_y(x, 0) = x^2 = 0$$

$$\therefore f_y(\Delta x, 0) = (\Delta x)^2 \quad \ldots (7)$$

Now,

$$\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{(0,0)} = f_{xy}(0, 0)$$

$$= \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 - 0}{\Delta x}$$

[From equation (6) and (7)]

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (\Delta x) = 0$$

$$\therefore \left[\frac{\partial^2 f}{\partial x \partial y}\right]_{(0,0)} = 0$$

And,

$$\left[\frac{\partial^2 f}{\partial y \partial x}\right]_{(0,0)} = f_{yx}(0, 0)$$

$$= \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y}$$

[From equations (4) and (5)]

$$= \lim_{\Delta y \to 0} \frac{0}{\Delta y}$$

$$= \frac{0}{0} = 0$$

$$\therefore \left[\frac{\partial^2 f}{\partial y \partial x}\right]_{(0,0)} = 0$$

$$\therefore \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0 \text{ at } (0, 0)$$

Hence proved.

Q28. Show that the function

$$\mathbf{f(x,y)=\begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}}$$ **is not differentiable at (0, 0).**

Answer : Dec.-16, Q14(a)

Given function is,

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$\Rightarrow \quad f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

$$f(0, 0) = 0$$

$$f(\Delta x, 0) = \frac{(\Delta x)^2 - 0^2}{(\Delta x)^2 + 0^2} = 1$$

$$\Rightarrow \quad f(\Delta x, 0) = 1$$

$$\Rightarrow \quad f(\Delta x, 0) = f(x, 0) = 1$$

$$f(0, \Delta y) = \frac{0^2 - (\Delta y)^2}{0^2 + (\Delta y)^2}$$

$$\Rightarrow \quad f(0, \Delta y) = -1$$

$$\Rightarrow \quad f(0, \Delta y) = f(0, y) = -1$$

(i) $$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\Rightarrow \quad f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} = \infty$$

$$\therefore \quad f_x(0, 0) = \infty$$

(ii) $$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$\Rightarrow \quad f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - (0, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{-1 - 0}{\Delta y} = \lim_{\Delta y \to 0} \frac{-1}{\Delta y} = -\infty$$

$$\therefore \quad f_y(0, 0) = -\infty$$

$$f_x(0, 0) \neq f_y(0, 0)$$

$\therefore$ f is not differentiable.

Q29. Show that the function $\mathbf{f(x,y)=\begin{cases} \frac{x^2+y^2}{x-y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}}$ **is not differentiable at (0,0).**

June-13, Q14(b)

OR

Show that $\mathbf{f(x, y) = \begin{cases} \frac{x^2+y^2}{x-y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}}$ **is not continuous at (0, 0).**

Answer : June-11, Q12(b)

Given that,

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x - y}$$

Let,

$$P_1 = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^2 + y^2}{x - y}\right]$$

$$= \lim_{y \to 0} \left[\lim_{x \to 0} \frac{x^2 + y^2}{x - y}\right] = \lim_{y \to 0} \left[\frac{0 + y^2}{0 - y}\right]$$

$$= \lim_{y \to 0} \left[-\frac{y^2}{y}\right] = \lim_{y \to 0} [-y]$$

$$= \lim_{y \to 0} [-0] = 0$$

$$\therefore \quad P_1 = 0 \qquad \ldots (1)$$

Let,

$$P_2 = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^2+y^2}{x-y}$$

$$= \lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+0}{x-0}\right] = \lim_{x\to 0}\left[\frac{x^2}{x}\right]$$

$$= \lim_{x\to 0}[x] = 0$$

$$\therefore \quad P_2 = 0 \qquad \ldots (2)$$

Let,

$$P_3 = \lim_{\substack{y\to mx\\ x\to 0}}\left[\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\lim_{y\to mx}\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+(mx)^2}{x-mx}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+m^2x^2}{x(1-m)}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2(1+m^2)}{x(1-m)}\right]$$

$$= \lim_{x\to 0}\left[\frac{x(1+m^2)}{(1-m)}\right] = \frac{0(1+m^2)}{(1-m)} = 0$$

$$\therefore \quad P_3 = 0 \qquad \ldots (3)$$

Let,

$$P_4 = \lim_{\substack{y\to mx^2\\ x\to 0}}\left[\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\lim_{y\to mx^2}\frac{x^2+y^2}{x-y}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+(mx^2)^2}{x-mx^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2+m^2x^4}{x-mx^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^2(1+m^2x^2)}{x(1-mx)}\right]$$

$$= \lim_{x\to 0}\left[\frac{x(1+m^2x^2)}{(1-mx)}\right]$$

$$= \frac{0[1+m^2(0)^2]}{[1-m(0)]} = 0$$

$$\therefore \quad P_4 = 0 \qquad \ldots (4)$$

Since the value of limit along any path is same, the limit exists and the limit value is zero.

Therefore, the $f(x,y)$ is continuous at origin only if $(x,y) = 0$ for $x = 0$ and $y = 0$ otherwise the function $f(x,y)$ is not continuous at (0,0).

3.2 PARTIAL DERIVATIVES

Q30. Explain the concept of partial derivatives.

Answer :

Partial derivatives

Consider a function, $u = f(x, y)$

Where, x, y – Two independent variables.

The derivative of function 'u' with respect to 'x' by keeping another variable 'y' as constant is known as partial derivative of 'u' with respect to y.

The partial derivative of a function with respect to x is denoted by $\frac{\partial u}{\partial x}$

Similarly, partial derivative of a function with respect to 'y' is denoted, by $\frac{\partial u}{\partial y}$

Second Order Partial Derivative

The derivative of a first order partial derivative $\left(i.e., \frac{\partial u}{\partial x}\right)$ is called second order partial derivative.

The second order partial derivative of $\frac{\partial u}{\partial x}$ with respect to 'x' is,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

Where, $u = f(x, y)$

Similarly, the second order partial derivative of $\frac{\partial u}{\partial y}$ with respect to 'y' is ,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

Differentiating a function $\frac{\partial u}{\partial x}$ with respect to 'y',

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

Differentiating a function $\frac{\partial u}{\partial y}$ with respect to 'x',

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

Q31. If x = r cos θ, y = r sin θ, find $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$

Answer :

Given equations are,

$x = r\cos\theta$

$y = r\sin\theta$

Squaring and adding the above equations,

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$\Rightarrow \quad x^2 + y^2 = r^2$

$\Rightarrow \quad r = \sqrt{x^2 + y^2}$ ... (1)

Differentiating equation (1) partially with respect to x ,

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} \times 2x$$

$$= \frac{x}{\sqrt{x^2 + y^2}} \quad \text{... (2)}$$

Differentiating equation (1) partially with respect to y,

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2}} \times 2y$$

$$= \frac{y}{\sqrt{x^2 + y^2}} \quad \text{... (3)}$$

Squaring and adding equations (2) and (3),

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \left[\frac{x}{\sqrt{x^2 + y^2}}\right]^2 + \left[\frac{y}{\sqrt{x^2 + y^2}}\right]^2$$

$$= \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} = \frac{x^2 + y^2}{x^2 + y^2}$$

$$= 1$$

$\therefore \quad \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$.

Q32. If $z = e^{ax+by} f(ax - by)$, then prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

Answer :

Given that,

$z = e^{ax + by} \cdot f(ax - by)$

Consider, $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$ ...(1)

Consider,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[e^{ax+by} \cdot f(ax - by)]$$

$$= e^{ax + by} \times f'(ax - by)(a) + f(ax - by) \times e^{ax + by}(a)$$

$\therefore \quad \frac{\partial z}{\partial x} = ae^{ax + by}[f'(ax - by) + f(ax - by)]$ ... (2)

Consider, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[e^{ax+by} \cdot f(ax-by)]$

$= e^{ax+by} \times f'(ax-by) \times (-b) + f(ax-by) \times e^{ax+by}(b)$

$\therefore \quad \frac{\partial z}{\partial y} = b.e^{ax+by}[-f'(ax-by) + f(ax-by)]$

Substituting equations (2) and (3) in equation (1), ... (3)

$\therefore \quad b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = bae^{ax+by}[f'(ax-by) + f(ax-by)] + abe^{ax+by}[-f'(ax-by) + f(ax-by)]$

$= abe^{ax+by}[f'(ax-by) + f(ax-by) - f'(ax-by) + f(ax-by)]$

$= abe^{ax+by}[2.f(ax-by)]$

$= 2abz \qquad [\because z = e^{ax+by} f(ax-by)]$

$\therefore \quad b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$

Q33. If $u = (x^2+y^2+z^2)^{\frac{-1}{2}}$ then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Answer :

Given that,

$u = (x^2+y^2+z^2)^{-1/2}$

Differentiating equation (1) partially with respect to x.

$\frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1}2x$

$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2x$

$\Rightarrow \quad \frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$

$\frac{\partial u^2}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]$

$= \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$

$= -\left[x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{\frac{-3}{2}-1}(2x) + (x^2+y^2+z^2)^{\frac{-3}{2}}\right]$

$= 3x^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$ ... (2)

Similarly,

$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$ ... (3)

And,

$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{\frac{-5}{2}} + (x^2+y^2+z^2)^{\frac{-3}{2}}$ ... (4)

Adding equations (2), (3) and (4),

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \left[3x^2(x^2+y^2+z^2)^{\frac{-5}{2}}+(x^2+y^2+z^2)^{\frac{-3}{2}}\right]+\left[3y^2(x^2+y^2+z^2)^{\frac{-5}{2}}+(x^2+y^2+z^2)^{\frac{-3}{2}}\right]$$
$$+\left[3z^2(x^2+y^2+z^2)^{\frac{-5}{2}}+(x^2+y^2+z^2)^{\frac{-3}{2}}\right]$$
$$= 3(x^2+y^2+z^2)^{\frac{-5}{2}}[x^2+y^2+z^2]+3(x^2+y^2+z^2)^{\frac{-3}{2}}$$
$$= 3(x^2+y^2+z^2)^{\frac{-3}{2}}[(x^2+y^2+z^2)^{-1}(x^2+y^2+z^2)+1]$$
$$= 3(x^2+y^2+z^2)^{\frac{-3}{2}}[1+1]$$
$$= 3(x^2+y^2+z^2)^{\frac{-3}{2}}(2)$$
$$= 6(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=6(x^2+y^2+z^2)^{\frac{-3}{2}}$$

3.3 TOTAL DIFFERENTIAL AND DIFFERENTIABILITY

Q34. Write a brief note on total derivatives.

Answer :

Let, $u = f(x, y, z)$, where x, y, z are function of a one independent variable 't' i.e., $x = x(t), y = y(t), z = z(t)$. Then considering the dependent variable 'f' as a function of single independent variable 't'. So, the derivative of function 'f' with respect to 't' is called as total derivative of 'f' which is given as,

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt} \qquad \ldots (1)$$

If y and z are functions of x, then f is a function of single independent variable 'x'. Hence, s equation (1) becomes,

$$\frac{df}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}$$

$$\therefore \quad \frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}$$

The above equation is called as chain rule of partial differentiation for one independent variable. This rule can be extented to functions of more than two independent variables.

Q35. Find $\frac{du}{dt}$ if $u = \tan^{-1}\left(\frac{y}{x}\right)$ and $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$.

Answer :

Given that,

$$u = \tan^{-1}\left(\frac{y}{x}\right) \text{ and}$$

$$x = e^t - e^{-t} \;;\; y = e^t + e^{-t}$$

Differentiating above equations above equations with respect to 't',

$$\frac{dx}{dt} = e^t + e^{-t} \;;\; \frac{dy}{dt} = e^t - e^{-t}$$

Differentiating 'u' partially with respect to x and y,

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{-x^2 y}{x^2+y^2}.\frac{1}{x^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{-y}{(x^2+y^2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$$

From the definition of total derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Substituting the corresponding values in above equation,

$$\frac{du}{dt} = \frac{-y}{x^2+y^2}(e^t + e^{-t}) + \frac{x}{x^2+y^2}(e^t - e^{-t})$$

Substituting 'x' and 'y' values in above equation,

$$\frac{du}{dt} = \frac{-(e^t+e^{-t})(e^t+e^{-t})}{(e^t-e^{-t})^2+(e^t+e^{-t})^2} + \frac{(e^t-e^{-t})(e^t-e^{-t})}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}$$

$$= \frac{1}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}\ [-(e^t+e^{-t})(e^t+e^{-t}) + (e^t-e^{-t})(e^t-e^{-t})]$$

$$= \frac{1}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}\ [-(e^t+e^{-t})^2 + (e^t-e^{-t})^2]$$

$$= \frac{-(e^{2t}+e^{-2t}+2)+(e^{2t}+e^{-2t}-2)}{e^{2t}+e^{-2t}-2+e^{2t}+e^{-2t}+2}$$

$$= \frac{-e^{2t}-e^{-2t}-2+e^{2t}+e^{-2t}-2}{2e^{2t}+2e^{-2t}}$$

$$= \frac{-4}{2(e^{2t}+e^{-2t})}$$

$$= \frac{-2}{e^{2t}+e^{-2t}}$$

$$= \frac{-2}{2\cosh 2t}$$

$$= \frac{-1}{\cosh 2t}$$

$$\therefore \quad \frac{du}{dt} = \frac{-1}{\cosh 2t}.$$

Q36. Find $\frac{du}{dt}$ as a total derivative if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$.

Answer :

Given that,

$$u = x^2 + y^2 + z^2$$

$$x = e^{2t}; y = e^{2t}\cos 3t; \quad z = e^{2t}\sin 3t$$

Differentiating above equations with respect to t,

$$\frac{dx}{dt} = 2e^{2t} \; ; \; \frac{dy}{dt} = 2\, e^{2t} \cos 3t - 3e^{2t} \sin 3t$$

$$\frac{dz}{dt} = 2\, e^{2t} \sin 3t + 3e^{2t} \cos 3t$$

Differentiating u partially with respect to x and y,

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial z} = 2z$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Substituting the corresponding values in above equation,

$$\frac{du}{dt} = 2x\,(2e^{2t}) + 2y\,[e^{2t}\,(2\cos 3t - 3\sin 3t)] + 2z\,[e^{2t}\,(2\sin 3t + 3\cos 3t)]$$

Substituting the values of x, y and z in above equation,

$$\frac{du}{dt} = 4\, e^{4t} + 4e^{4t} \cos^2 3t - 6e^{4t} \cos 3t \sin 3t + 4e^{4t} \sin^2 3t + 6e^{4t} \sin 3t \cos 3t$$

$$\Rightarrow \quad \frac{du}{dt} = 4e^{4t} + 4e^{4t}\,[\cos^2 3t + \sin^2 3t] = 4e^{4t} + 4e^{4t}\,(1)$$

$$\therefore \quad \frac{du}{dt} = 8\, e^{4t}.$$

Q37. If $u = xy + yz + zx$ where $x = \frac{1}{t}$, $y = e^t$ and $z = e^{-t}$ find $\frac{du}{dt}$.

Answer :

Given that,

$$u = xy + yz + zx \qquad \ldots (1)$$

$$x = \frac{1}{t} \qquad \ldots (2)$$

$$y = e^t \qquad \ldots (3)$$

$$z = e^{-t} \qquad \ldots (4)$$

Differentiating equation (1) partially with respect to 'x', 'y', and 'z' respectively

$$\frac{\partial u}{\partial x} = y + z$$

$$\frac{\partial u}{\partial y} = x + z$$

$$\frac{\partial u}{\partial z} = y + x$$

Differentiating equations (2), (3) and (4), partially with respect to 't',

$$\frac{\partial x}{\partial t} = \frac{-1}{t^2} \; ; \; \frac{\partial y}{\partial t} = e^t$$

$$\frac{\partial z}{\partial t} = -\, e^{-t}$$

From the definition of total derivatives,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} \quad \ldots (5)$$

Substituting the corresponding values in equation (5),

$$\frac{du}{dt} = (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

$$= \frac{-(y+z)}{t^2} + e^t(x+z) - \frac{(y+x)}{e^t}$$

$$\Rightarrow \quad \frac{du}{dt} = \frac{-(y+z)}{t^2} + e^t(x+z) - \frac{(y+x)}{e^t} \quad \ldots (6)$$

Substituting equations (2), (3) and (4) in equation (6),

$$\frac{du}{dt} = \frac{-(e^t + e^{-t})}{t^2} + e^t\left(\frac{1}{t} + e^{-t}\right) - \frac{\left(e^t + \frac{1}{t}\right)}{e^t}$$

$$= \frac{-2\cosh t}{t^2} + \frac{e^t}{t} + 1 - 1 - \frac{e^{-t}}{t}$$

$$= \frac{-2\cosh t}{t^2} + \frac{2\sinh t}{t}$$

$$\therefore \quad \frac{du}{dt} = \frac{-2\cosh t}{t^2} + \frac{2\sinh t}{t}.$$

3.4 DERIVATIVES OF COMPOSITE AND IMPLICIT FUNCTIONS (CHAIN RULE), CHANGE OF VARIABLES

Q38. Write about,

(i) Derivatives of composite function

(ii) Derivatives of implicit function.

Answer :

(i) Derivatives of Composite Function

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \Psi(t)$, then 'u' is called a composite function of two variables and is expressed as,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

(ii) Derivatives of Implicit Function

A function of the form $f(x, y) = C$ is known as implicit function.

Where,

C - Constant.

The expression for the first differential coefficient of an implicit function is,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Q39. If $u = \sin^{-1}(x - y)$, $x = 3t$ and $y = 4t^3$, then show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Answer :

Given that,

$$u = \sin^{-1}(x - y) \quad \ldots (1)$$

$$x = 3t \quad \ldots (2)$$

$$y = 4t^3 \quad \ldots (3)$$

The above equations represent a composite function of t and is expressed as,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \quad \ldots (4)$$

Partially differentiating equation (1), with respect to 'x',

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \quad \ldots (5)$$

Partially differentiating equation (1) with respect to 'y',

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(-1) \quad \ldots (6)$$

From equations (2) and (3),

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = 12t^2 \quad \ldots (7)$$

Substituting equations (5), (6) and (7) in equation (4),

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}.3 + \frac{1}{\sqrt{1-(x-y)^2}}(-1)(12t^2)$$

$$= \frac{3}{\sqrt{1-(x-y)^2}} + \frac{(-12t^2)}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$\therefore \quad \frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}$$

Substituting the values of 'x' and 'y' in above equation,

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(9t^2+16t^6-24t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)}(1-4t^2)}$$

$$= \frac{3}{\sqrt{(1-t^2)}}$$

$$\therefore \frac{du}{dt} = \frac{3}{\sqrt{(1-t^2)}}.$$

Q40. Using implicit differentiation, obtain $\frac{dy}{dx}$ when $f(x, y) = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right) = 0$.

Answer :

Given that,

$$f(x, y) = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right) \quad \text{... (1)}$$

The implicit function is expressed as,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Partially differentiating equation (1) with respect to x and y, ... (2)

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} + \frac{-\left(\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} \quad ; \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} \quad ; \quad \frac{\partial f}{\partial y} = \frac{2y+x}{x^2+y^2}$$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{2x-y}{x^2+y^2}$$

Substituting the corresponding values in equation (2),

$$\frac{dy}{dx} = \frac{-\frac{2x-y}{x^2+y^2}}{\frac{2y+x}{x^2+y^2}} = \frac{y-2x}{2y+x}$$

$$\therefore \frac{dy}{dx} = \frac{y-2x}{2y+x}$$

Q41. Given the transformations $u = e^x \cos y$ and $v = e^x \sin y$ and that ϕ is a function of u and v and also of x and y. Prove that $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right)$.

Answer :

Given transformations are,

$u = e^x \cos y$; $v = e^x \sin y$

Differentiating above equations with respect to x and y,

$$\frac{\partial u}{\partial x} = e^x \cos y \; ; \; \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \; ; \; \frac{\partial v}{\partial y} = e^x \cos y$$

Consider, $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}$ [$\because$ From chain rule]

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} e^x \cos y + \frac{\partial \phi}{\partial v} e^x \sin y$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} \quad \ldots (1)$$

$$\frac{\partial}{\partial x} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \quad \ldots (2)$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial x}\right]$$

$$= \left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right)$$ [$\because$ From equations (1) and (2)]

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} = u^2\frac{\partial^2 \phi}{\partial u^2} + uv\frac{\partial^2 \phi}{\partial u\,\partial v} + uv\frac{\partial^2 \phi}{\partial u\,\partial v} + v^2\frac{\partial^2 \phi}{\partial v^2} \quad \ldots (3)$$

Consider, $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}$

$$= \frac{\partial \phi}{\partial u}\left(-e^x \sin y\right) + \frac{\partial \phi}{\partial v}\left(e^x \cos y\right)$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial y} = -v\frac{\partial \phi}{\partial u} + u\frac{\partial \phi}{\partial v} \quad \ldots (4)$$

$$\Rightarrow \quad \frac{\partial}{\partial y} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} \quad \ldots (5)$$

Differentiating equation (4) partially with respect to 'y',

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$$

$$= \left(-v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right)\left(-v\frac{\partial \phi}{\partial u} + u\frac{\partial \phi}{\partial v}\right)$$ [$\because$ From equations 4 and 5]

$$\therefore \quad \frac{\partial^2 \phi}{\partial y^2} = v^2\frac{\partial^2 \phi}{\partial u^2} - uv\frac{\partial \phi}{\partial u\,\partial v} - uv\frac{\partial \phi}{\partial u\,\partial v} - u^2\frac{\partial^2 \phi}{\partial v^2} \quad \ldots (6)$$

Adding equations (3) and (6),

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = u^2\frac{\partial^2 \phi}{\partial u^2} + uv\frac{\partial^2 \phi}{\partial u\,\partial v} + uv\frac{\partial^2 \phi}{\partial u\,\partial v} + v^2\frac{\partial^2 \phi}{\partial v^2} + v^2\frac{\partial^2 \phi}{\partial u^2} - uv\frac{\partial \phi}{\partial u\,\partial v} - uv\frac{\partial \phi}{\partial u\,\partial v} + u^2\frac{\partial^2 \phi}{\partial v\ 2}$$

$$= \frac{\partial^2 \phi}{\partial u^2}(u^2 + v^2) + \frac{\partial^2 \phi}{\partial v^2}(v^2 + u^2)$$

$$= (u^2 + v^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right).$$

Q42. If $u = \log(x^2 + y^3 + z^3 - 3xyz)$. Show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}$.

Answer :

Given function is,

$$u = \log(x^2 + y^3 + z^3 - 3xyz) \quad \ldots (1)$$

Differentiating equation (1) partially with respect to 'x', 'y' and 'z'

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3x^2 - 3yz) \quad \text{... (2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3y^2 - 3xz) \quad \text{... (3)}$$

$$\frac{\partial u}{\partial z} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3z^2 - 3xy) \quad \text{... (4)}$$

Adding equations (2), (3) and (4),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3y^2 + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x + y + z} \quad \text{... (5)}$$

Differentiating equation (5) partially with respect to 'x', 'y' and 'z'

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} \quad \text{... (6)}$$

$$\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} \quad \text{... (7)}$$

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} \quad \text{... (8)}$$

Adding equations (6), (7) and (8),

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u\right] = \frac{-3}{(x + y + z)^2} + \left(\frac{-3}{(x + y + z)^2}\right) + \left(\frac{-3}{(x + y + z)^2}\right)$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}$$

$$\therefore \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}.$$

Q43. Explain change of variables in partial differentiation.

Answer :

Let u be a function of x and y given as,

$$u = f(x, y)$$

Where, x, y are functions of s, t

$$x = \phi(s, t)$$

$$y = \psi(s, t)$$

If t is considered as a constant then, x, y, u are the functions of 's' only.

From chain rule,

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

The ordinary derivatives are replaced by partial derivatives because x, y are functions of variables s and t.

$$\Rightarrow \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \quad \text{... (1)}$$

If s is considered as a constant then x, y, u will be functions of t only

$$\Rightarrow \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad \text{... (2)}$$

Equations (1) and (2) represent simultaneous equations in $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

The values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are obtained in terms of $\frac{\partial^2}{\partial s}, \frac{\partial u}{\partial t}$, u, s, t by solving equations (1) and (2)

Hence, the variables x, y are changed to s, t.

If the variables s and t are expressed in terms of x and y i.e.,

$s = \xi(x, y)$ and $t = \eta(x, y)$ then the following formulae are obtained

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

The above two equations are solved to obtain the values of $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$

Hence, the variables s, t are changed to x, y.

Q44. If $Z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that, $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$.

Answer :

Given that,

$$z = f(x, y)$$

$$x = e^{2u} + e^{-2v},\ y = e^{-2u} + e^{2v}$$

Differentiating above equations partially with respect to u and v,

$$\frac{\partial x}{\partial u} = 2e^{2u} \qquad \frac{\partial y}{\partial u} = -2e^{-2u}$$

$$\frac{\partial x}{\partial v} = -2e^{-2v} \qquad \frac{\partial y}{\partial v} = 2e^{2v}$$

From Chain rule,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\therefore \quad \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}(2e^{2u}) + \frac{\partial f}{\partial y}(-2e^{-2u}) \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}(-2e^{-2v}) + \frac{\partial f}{\partial y}(2e^{2v}) \qquad \text{... (2)}$$

Subtracting equation (2) from equation (1),

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}(2e^{2u} + 2e^{-2v}) + \frac{\partial f}{\partial y}(-2e^{-2u} - 2e^{2v})$$

$$\Rightarrow \quad \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}$$

$$\therefore \quad \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Q45. If x = u + v + w, y = uv + vw + wu, z = uvw, and f is a function of x, y, z then show that,

$$\mathbf{u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}.}$$

Answer :

Given that,

f is a function of x, y, z

$x = u + v + w \qquad y = uv + vw + wu \qquad z = uvw$

Differentiating above equations partially with respect to u, v and w,

$$\frac{\partial x}{\partial u}=1 \quad ; \quad \frac{\partial y}{\partial u}=v+w \quad ; \quad \frac{\partial z}{\partial u}=vw$$

$$\frac{\partial x}{\partial v}=1 \quad ; \quad \frac{\partial y}{\partial v}=u+w \quad ; \quad \frac{\partial z}{\partial v}=uw$$

$$\frac{\partial x}{\partial w}=1 \quad ; \quad \frac{\partial y}{\partial w}=u+v \quad ; \quad \frac{\partial z}{\partial w}=uv$$

From chain rule,

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial u}$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}(1)+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}(vw)$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial v}$$

$$\Rightarrow \quad \frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}(1)+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}(uw)$$

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}.\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial w}$$

$$\Rightarrow \quad \frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}(1)+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}(uv)$$

Consider $u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}$

$$= u\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}(vw)\right)+v\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}(uw)\right)+w\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}(uv)\right)$$

$$= u\frac{\partial f}{\partial x}+u(v+w)\frac{\partial f}{\partial y}+uvw\frac{\partial f}{\partial z}+v\frac{\partial f}{\partial x}+v(u+w)\frac{\partial f}{\partial y}+uvw\frac{\partial f}{\partial z}+w\frac{\partial f}{\partial x}+w(u+v)\frac{\partial f}{\partial y}+uvw\frac{\partial f}{\partial z}$$

$$= (u+v+w)\frac{\partial f}{\partial x}+2(uv+vw+wu)\frac{\partial f}{\partial y}+3uvw\frac{\partial f}{\partial z}$$

$$= x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}$$

$$\therefore \quad u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w}=x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}.$$

Q46. If z = f(x, y), where x = e^u cos v and y = e^u sin v, then show that $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$

Answer :

Given functions are,

$z = f(x, y)$

$x = e^u \cos v$ ... (1)

$y = e^u \sin v$ ... (2)

Differentiating equation (1) partially with respect to u and v,

$$\frac{\partial x}{\partial u} = e^u \cos v, \quad \frac{\partial x}{\partial v} = -e^u \sin v$$

Differentiating equation (2) partially with respect to u and v,

$$\frac{\partial y}{\partial u} = e^u \sin v, \quad \frac{\partial y}{\partial v} = e^u \cos v$$

From chain rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \text{... (3)}$$

Substituting the corresponding values in equation (3),

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)$$

Multiplying on both sides by 'y',

$$y\frac{\partial z}{\partial u} = y\left[\frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)\right]$$

$$= e^u \sin v\left[e^u \cos v\frac{\partial z}{\partial x} + e^u \sin v\frac{\partial z}{\partial y}\right] \qquad [\because \text{ From equation (2)}]$$

$$\therefore \quad y\frac{\partial z}{\partial u} = e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\sin^2 v\frac{\partial z}{\partial y} \quad \text{... (4)}$$

Similarly,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \text{... (5)}$$

Substituting the corresponding values in equation (5),

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v)$$

Multiplying on the sides by 'x',

$$x\frac{\partial z}{\partial v} = x\left[-e^u \sin v\frac{\partial z}{\partial x} + e^u \cos v\frac{\partial z}{\partial y}\right]$$

$$= e^u \cos v\left[-e^u \sin v\frac{\partial z}{\partial x} + e^u \cos v\frac{\partial z}{\partial y}\right] \qquad [\because \text{ From equation (1)}]$$

$$\therefore \quad x\frac{\partial z}{\partial v} = -e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\cos^2 v\frac{\partial z}{\partial y} \quad \text{... (5)}$$

Adding equations (4) and (5),

$$x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = -e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\cos^2 v\frac{\partial z}{\partial y} + e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\sin^2 v\frac{\partial z}{\partial y}$$

$$= e^{2u}(\sin^2 v + \cos^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y}$$

$$\therefore \; x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$$

3.5 JACOBIAN

Q47. If x + y + z = u, y + z = uv, z = uvw show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

Answer :

Given functions are,

$$u = x + y + z \quad \ldots (1)$$

$$uv = y + z \quad \ldots (2)$$

$$uvw = z \quad \ldots (3)$$

Substituting the value of z from equation (3), in equation (1),

$$uv = y + uvw$$

$$\Rightarrow \quad y = uv - uvw$$

$$\Rightarrow \quad y = uv\,[1 - w] \quad \ldots (4)$$

Substituting the values of z and y from equations (3) and (4) in equation (1),

$$u = x + uv\,[1 - w] + uvw$$

$$\Rightarrow \quad u = x + uv - uvw + uvw$$

$$\Rightarrow \quad u = x + uv$$

$$\Rightarrow \quad x = u - uv$$

$$\Rightarrow \quad x = u(1 - v) \quad \ldots (5)$$

Differentiating equation (5) with respect to 'u',

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}[u(1-v)]$$

$$\Rightarrow \quad \frac{\partial x}{\partial u} = \frac{\partial}{\partial u}[u - uv] = 1 - v$$

Differentiating above equation with respect to 'v',

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}[u - uv] = 0 - u$$

$$\Rightarrow \quad \frac{\partial x}{\partial v} = -u$$

Differentiating above equation with respect to 'w',

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}[u - uv]$$

$$\frac{\partial x}{\partial w} = 0$$

Differentiating equation 4 with respect to u, v and w,

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}[uv(1-w)] = \frac{\partial}{\partial u}[uv - uvw] = v - vw$$

$$\Rightarrow \quad \frac{\partial y}{\partial u} = v[1 - w]$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}[uv - uvw]$$

$$= u - uw$$

$$\Rightarrow \quad \frac{\partial y}{\partial v} = u[1 - w]$$

$$\frac{\partial y}{\partial w} = \frac{\partial}{\partial w}[uv - uvw]$$

$$= 0 - uv$$

$$\Rightarrow \quad \frac{\partial y}{\partial w} = -uv$$

Differentiating equation (3) with respect to u, v and w,

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}[uvw]$$

$$\Rightarrow \quad \frac{\partial z}{\partial u} = vw$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}[uvw]$$

$$\Rightarrow \quad \frac{\partial z}{\partial v} = uw$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}[uvw]$$

$$\Rightarrow \quad \frac{\partial z}{\partial w} = uv$$

The Jacobian of x, y, z is given as,

$$J(x, y, z) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \ldots (6)$$

Substituting the corresponding values in equation (6),

$$J(x, y, z) = \begin{vmatrix} (1-v) & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)[(u(1-w)\,(uv) - (uw \times -uv))]$$

$$-(-u)\,[v(1-w)\,(uv) - (vw \times -uv)] + 0$$

$$= (1-v)[u(uv - uvw) + u^2vw] + u[v(uv - uvw) + uv^2w]$$

$$= (1-v)[u^2v - u^2vw + u^2vw] + u[uv^2 - uv^2w + uv^2w]$$

$$= (1-v)[u^2v] + u[uv^2]$$

$$= (1-v)(u^2v) + u^2v^2$$

$$= u^2v[1 - v + v]$$

$$= u^2v[1 + 0]$$

$$= u^2v$$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

Q48. Prove that $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1}(x) + \sin^{-1}(y)$ are functionally dependent and find the relation between them.

Answer :

Given functions are,

$$u = x\sqrt{1-y^2} + y\sqrt{1-x^2} \quad \ldots (1)$$

$$v = \sin^{-1}(x) + \sin^{-1}(y) \quad \ldots (2)$$

u and v are said to be functionally dependent if they satisfy the following condition,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

Partially differentiating equation (1) with respect to 'x',

$$\frac{\partial u}{\partial x} = \sqrt{1-y^2} - 2xy\frac{1}{2\sqrt{1-x^2}}$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

Partially differentiating equation (1) with respect to 'y',

$$\frac{\partial u}{\partial y} = -2yx\frac{1}{2\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

Partially differentiating equation (2) with respect to 'x',

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-x^2}} + 0$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Partially differentiating equation (2) with respect to 'y',

$$\frac{\partial v}{\partial y} = 0 + \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= \left[\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right]\left[\frac{1}{\sqrt{1-y^2}}\right] - \left[\frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right]\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \left[1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right] - \left[\frac{-xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1\right]$$

$$= 1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} - 1$$

$$= 0$$

$\therefore$ u, v are functionally dependent

$$v = \sin^{-1}x + \sin^{-1}y$$

$$= \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$\left[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\right]$$

$$= \sin^{-1}(u)$$

$\Rightarrow$ $\sin v = u$

$\therefore$ Relation between u and v is $\sin v = u$.

Q49. Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 if $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$.

OR

If $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$ prove that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$.

Answer :

Given that,

$$y_1 = \frac{x_2\,x_3}{x_1} \quad \ldots (1)$$

$$y_2 = \frac{x_3\,x_1}{x_2} \quad \ldots (2)$$

$$y_3 = \frac{x_1\,x_2}{x_3} \quad \ldots (3)$$

Partially differentiating equations (1), (2) and (3) with respect to 'x_1'

$$\frac{\partial y_1}{\partial x_1} = x_2\,x_3\left(\frac{-1}{x_1^2}\right) = -\frac{x_2\,x_3}{x_1^2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$$

Partially differentiating equations (1), (2) and (3) with respect to 'x_2'

$$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$$

$$\frac{\partial y_2}{\partial x_2} = x_3x_1\left(-\frac{1}{x_2^2}\right) = -\frac{x_1x_3}{x_2^2}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$$

Partially differentiating equation (1), (2) and (3) with respect to 'x_3'

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_3}{\partial x_3} = x_1 x_2\left(\frac{-1}{x_3^2}\right) = \frac{-x_1 x_2}{x_3^2}$$

$$\therefore \quad \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$\Rightarrow \quad \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\left(\frac{-x_1 x_3}{x_2^2}\right)\left(\frac{-x_1 x_2}{x_3^2}\right) - \left(\frac{x_1}{x_2}\right)\left(\frac{x_1}{x_3}\right)\right] - \frac{x_3}{x_1}\left[\frac{x_3}{x_2}\left(\frac{-x_1 x_2}{x_3^2}\right) - \left(\frac{x_1}{x_2}\right)\left(\frac{x_2}{x_3}\right)\right] + \frac{x_2}{x_1}\left[\left(\frac{x_3}{x_2}\right)\left(\frac{x_1}{x_3}\right) - \left(\frac{-x_1 x_3}{x_2^2}\right)\left(\frac{x_2}{x_3}\right)\right]$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\frac{x_1^2}{x_2 x_3} - \frac{x_1^2}{x_2 x_3}\right] - \frac{x_3}{x_1}\left[\frac{-x_1}{x_3} - \frac{x_1}{x_3}\right] + \frac{x_2}{x_1}\left[\frac{x_1}{x_2} + \frac{x_1}{x_2}\right]$$

$$= -\frac{x_2 x_3}{x_1^2}[0] - \frac{x_3}{x_1}\left[-\frac{2x_1}{x_3}\right] + \frac{x_2}{x_1}\left[\frac{2x_1}{x_2}\right]$$

$$= 2 + 2 = 4$$

$$\therefore \quad \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4.$$

Q50. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

Answer :

Given that,

$$x = r\sin\theta\cos\phi \quad \text{... (1)}$$

$$y = r\sin\theta\sin\phi \quad \text{... (2)}$$

$$z = r\cos\theta \quad \text{... (3)}$$

Partially differentiating equations (1), (2) and (3) with respect to 'r'.

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \;;\; \frac{\partial y}{\partial r} = \sin\theta\sin\phi \;;\; \frac{\partial z}{\partial r} = \cos\theta$$

Partially differentiating equations (1), (2) and (3) with respect to 'θ'.

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \;;\; \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \;;\; \frac{\partial z}{\partial \theta} = -r\sin\theta$$

Partially differentiating equations (1), (2) and (3) with respect to 'ϕ'.

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \;;\; \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \;;\; \frac{\partial y}{\partial \phi} = 0$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi\ [r\cos\theta\sin\phi(0) + r\sin\theta(r\sin\theta\cos\phi)] - r\cos\theta\cos\phi\ [\sin\theta\sin\phi(0) - \cos\theta(r\sin\theta\cos\phi)]$

$- r\sin\theta\sin\phi\ [\sin\theta\sin\phi(-r\sin\theta) - \cos\theta(r\cos\theta\sin\phi)]$

$= \sin\theta\cos\phi\ [r\sin\theta(r\sin\theta\cos\phi) - [r\cos\theta\cos\phi]$

$[-\cos\theta(r\sin\theta\cos\phi)] - r\sin\theta\sin\phi\ [-r\sin\theta(\sin\theta\sin\phi) - \cos\theta(r\cos\theta\sin\phi)]$

$= \sin\theta\cos\phi[r^2\sin^2\theta\cos\phi] - r\cos\theta\cos\phi[-r\sin\theta\cos\theta\cos\phi] - r\sin\theta\sin\phi\ [-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$

$= r^2\sin^3\theta\ [\cos^2\phi + \sin^2\phi] + r^2\sin\theta\cos^2\theta\ [\cos^2\phi + \sin^2\phi]$

$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta \qquad [\because \sin^2\phi + \cos^2\phi = 1]$

$= r^2\sin\theta\ [\sin^2\theta + \cos^2\theta]$

$= r^2\sin\theta$

$\therefore \quad \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$

3.6 HIGHER ORDER PARTIAL DERIVATIVES

Q51. Explain the concept of finding the higher order derivatives of second and third order with examples.

Answer :

Higher Order Derivatives

The derivatives of the principle derivative $\dfrac{dy}{dx}$ are known as higher order derivatives.

(i) Second Order Derivative

The derivative of first order derivative $\left(\text{i.e.,}\dfrac{dy}{dx}\right)$ is known as second derivative or second order derivative.

$$\text{i.e.,}\ \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dx^2}$$

The different notations of second order derivative are,

1. D^2y
2. y^2
3. y''
4. f''

Example

If $y = x^3$, then $\dfrac{d^2y}{dx^2}$

Given that,

$y = x^3$ `...(1)

Differentiating equation (1) with respect to 'x',

$$\frac{dy}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(x^3) = 3x^{3-1} \qquad \left[\because \frac{d}{dx}x^n = nx^{n-1}\right]$$

$= 3x^2$ ...(2)

Differentiating equation (2) with respect to 'x',

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\ 3x^2$$

$$= 3\ \frac{d}{dx}x^2 = 3\ (2x^{2-1})$$

$$\frac{d^2y}{dx^2} = 6x$$

$$\therefore \frac{d^2y}{dx^2} = 6x$$

(ii) Third Order Derivative

The derivative of second order derivative $\left(\text{i.e.,}\frac{d^2y}{dx^2}\right)$ is known as third order derivative.

i.e., $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

The different notations of third order derivatives are,

1. D^3y
2. y_3
3. y'''
4. f'''

Example

If $y = x^4$, then $\frac{d^3y}{dx^3}$

Given that,

$y = x^4$ ... (1)

Differentiating equation (1) with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}\ x^4$$
$$= 4x^{4-1}$$

$\Rightarrow \quad \frac{dy}{dx} = 4x^3$ ... (2)

Differentiating equation (2) with respect to 'x',

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3)$$
$$= 4 \times 3x^{3-1}$$

$$\frac{d^2y}{dx^2} = 12x^2 \quad \text{... (3)}$$

Differentiating equation (3) with respect to 'x',

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(12x^2)$$
$$\frac{d^3y}{dx^3} = 12\ (2x^{2-1})$$
$$= 24x$$
$$\therefore \frac{d^3y}{dx^3} = 24x$$

Q52. If $y = ae^x + be^{-x}$, show that $\frac{d^2y}{dx^2} - y = 0$.

Answer :

Given equation is,

$y = ae^x + be^{-x}$ ... (1)

Differentiating equation (1) on both sides with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}\left[ae^x + be^{-x}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = a\ [e^x] + b\ (e^{-x}(-1))$$

$$\Rightarrow \quad \frac{dy}{dx} = ae^x - be^{-x} \qquad \text{... (2)}$$

Again differentiating equation (2) with respect to x,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[ae^x - be^{-x}] \qquad \left[\begin{array}{l}\frac{d}{dx}(e^x) = e^x \\ \frac{d}{dx}(e^{-x}) = e^{-x}(-1)\end{array}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = a[e^x] - b\ [e^{-x}(-1)]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = ae^x + be^{-x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = y \qquad [\because \text{ From equation (1)}]$$

$$\therefore \frac{d^2y}{dx^2} - y = 0$$

Q53. If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

Answer :

Given function is,

$$y = a\cos(\log x) + b\sin(\log x) \qquad \text{... (1)}$$

Differentiating equation (1) on both sides with respect to x,

$$\frac{dy}{dx} = a\frac{d}{dx}[\cos(\log x)] + b\frac{d}{dx}[\sin(\log x)]$$

$$\Rightarrow \quad \frac{dy}{dx} = a\frac{d}{dx}[\cos(\log x)]\frac{d}{dx}(\log x) + b\cos(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = -a\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x) \qquad \text{...(2)}$$

Differentiating equation (2) on both sides with respect to x,

$$x\frac{d}{dx}\left[\frac{dy}{dx}\right] + \frac{dy}{dx} \times \frac{d}{dx}(x) = -a\cos(\log x) \times \frac{1}{x} - b\sin(\log x) \times \frac{1}{x}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times (1) = -\left[\frac{a\cos(\log x) + b\sin(\log x)}{x}\right]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \qquad [\because \text{From equation (1)}]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \qquad \text{...(3)}$$

Since, $\frac{dy}{dx} = y_1$ and $\frac{d^2y}{dx^2} = y_2$

Thus, equation (3) reduces to,

$$x^2y_2 + xy_1 + y = 0$$

3.7 TAYLOR'S SERIES OF FUNCTIONS OF TWO VARIABLES

Q54. Prove Taylor's series for the functions of two variables.

Answer :

Let, $f(x + h, y + k)$ be a function of one variable i.e., x.

Then according to the Taylor's theorem of a single variable,

$$f(x+h, y+k) = f(x, y+k) + \frac{h}{1!}\frac{\partial f(x,y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x,y+k)}{\partial x^2} + \ldots. \quad \ldots (1)$$

If $f(x, y + k)$ is a function of y, then,

$$f(x, y+k) = f(x,y) + \frac{k}{1!}\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots. + \frac{h}{1!}\frac{\partial}{\partial x}\left[f(x,y) + \frac{k}{1!}\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots \right] \quad \ldots (2)$$

Substituting equation (2) in equation (1),

$$f(x+h, y+k) = f(x,y) + \frac{k}{1!}\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots. + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x,y) + \frac{k}{1!}\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots \right]$$

$$= f(x,y) + \frac{k}{1!}\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \frac{h}{1!}\frac{\partial f(x,y)}{\partial x} + hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{hk^2}{2!}\frac{\partial^3 f(x,y)}{\partial x \partial y^2} + \frac{h^2}{2!}\frac{\partial^2 f(x,y)}{\partial x^2}$$

$$+ \frac{h^2 k}{2!}\frac{\partial^3 f(x,y)}{\partial x^2 \partial y} + \frac{h^2 k^2}{2!2!}\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} + \ldots..$$

Neglecting higher order derivatives,

$$f(x+h, y+k) = f(x,y) + \frac{k}{1!}\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \frac{h}{1!}\frac{\partial f(x,y)}{\partial x} + hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{h^2}{2!}\frac{\partial^2 f(x,y)}{\partial x^2} + \ldots.$$

$$= f(x,y) + \frac{h}{1!}\frac{\partial f(x,y)}{\partial x} + \frac{k}{1!}\frac{\partial f(x,y)}{\partial y} + \frac{1}{2!}\left[h^2\frac{\partial^2 f(x,y)}{\partial x^2} + k^2\frac{\partial^2 f(x,y)}{\partial y^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x \partial y} \right] + \ldots.$$

$$= f(x,y) + \left[\frac{h}{1!} f_x(x,y) + \frac{k}{1!} f_y(x,y) \right] + \frac{1}{2!}\left[h^2 f_{xx}(x,y) + k^2 f_{yy}(x,y) + 2hk f_{xy}(x,y) \right] + \ldots$$

Let,

$x = a$ and $y = b$, then,

$$f(a+h, b+k) = f(a,b) + \left[\frac{h}{1!} f_x(a,b) + \frac{k}{1!} f_y(a,b) \right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hk f_{xy}(a,b) \right] + \ldots.$$

Substituting $a + h = x$ and $b + k = y$ such that $h = x - a$ and $k = y - b$ in the above equation,

$$f(x,y) = f(a,b) + \left[\frac{(x-a)}{1!} f_x(a,b) + \frac{(y-b)}{1!} f_y(a,b) \right] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)] + \ldots..$$

The above equation is the Taylor's theorem which is used to expand $f(x, y)$ in (a, b) i.e., two variables.

Q55. Obtain the Taylor series expansion of the function $f(x, y) = e^{2x+y}$ about (0, 0) upto third degree terms.

Answer : June/July-17, Q14(b)

Given function is,

$$f(x, y) = e^{2x+y} \quad \ldots (1)$$

$$f(0,0) = e^{2(0)+0} = 1$$

Differentiating equation (1) partially with respect to x and y.

$f_y(x, y) = e^{2x+y}$ $\quad f_y(0,0) = e^{0+0} = 1$

$f_{yy}(x, y) = e^{2x+y}$ $\quad f_{yy}(0,0) = e^{0+0} = 1$

$f_x(x, y) = 2e^{2x+y}$ $\quad f_x(0,0) = 2e^{0+0} = 2$

$f_{xx}(x, y) = 4e^{2x+y}$ $\quad f_{xx}(0,0) = 4e^{0+0} = 4$

$f_{yx}(x, y) = 2e^{2x+y}$ $\quad f_{yx}(0,0) = 2e^{0+0} = 2$

$f_{xy}(x, y) = 2e^{2x+y}$ $\quad f_{xy}(0,0) = 2e^{0+0} = 2$

$f_{xyy}(x, y) = 2e^{2x+y}$ $\quad f_{xyy}(0,0) = 2e^{0+0} = 2$

$f_{xxy}(x, y) = 4e^{2x+y}$ $\quad fx_{xy}(0,0) = 4e^{0+0} = 2$

$f_{xxx}(x, y) = 8e^{2x+y}$ $\quad f_{xxx}(0,0) = 8e^{0+0} = 8$

$f_{yyy}(x, y) = e^{2x+y}$ $\quad f_{yyy}(0,0) = e^{0+0} = 1$

Taylor's series is given as,

$$f(x, y) = f(x_0, y_0) + (x - x_0)\, f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) + \frac{1}{2!}[(x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0) f_{xy}(x_0, y_0) +$$
$$(y - y_0)^2 f_{yy}(x_0, y_0) + \frac{1}{3!}[(x - x_0)^3 f_{xxx}(x_0, y_0) + 3(x - x_0)^2 (y - y_0) f_{xxy}(x_0, y_0)$$
$$+ 3(x - x_0)(y - y_0)^2 f_{xyy}(x_0, y_0) + (y - y_0)^3 f_{yyy}(x_0, y_0)]$$
$$= f(0, 0) + (x - 0) f_x(0, 0) + (y - 0) f_y(0, 0) + \frac{1}{2!}[(x - 0)^2 f_{xx}(0, 0) +$$
$$2(x - 0)(y - 0) f_{xy}(0, 0) + (y - 0)^2 f_{yy}(0, 0)] + \frac{1}{3!}[(x - 0)^3 f_{xxx}(0, 0)$$
$$+ 3(x - 0)^2 (y - 0) f_{xxy}(0, 0) + 3(x - 0)(y - 0)^2 f_{xyy}(0, 0) + (y - 0)^3 f_{yyy}(0, 0)] \quad \ldots (2)$$

Substituting the corresponding values in equation (2),

$$= 1 + x(2) + y(1) + \frac{1}{2}[x^2(4) + 2xy(2) + y^2(1)] + \frac{1}{6}[x^3(8) + 3x^2y(4) + 3xy^2(2) + y^3(1)]$$
$$= 1 + 2x + y + \frac{1}{2}[4x^2 + 4xy + y^2] + \frac{1}{6}[8x^3 + 12x^2y + 6xy^2 + y^3]$$
$$= 1 + 2x + y + 2x^2 + 2xy + \frac{y^2}{2} + \frac{4}{3}x^3 + 2x^2y + xy^2 + \frac{y^3}{6}$$

$\therefore \quad e^{2x+y} = 1 + 2x + y + 2x^2 + 2xy + \frac{y^2}{2} + \frac{4}{3}x^3 + 2x^2y + xy^2 + \frac{y^3}{6}$.

Q56. Obtain the Taylor's series expansion of $x^3 + y^3 + xy^2$ in terms of powers of $(x - 1)$ and $(y - 2)$ up to third degre terms.

Answer :

Given function is,

$$f(x, y) = x^3 + y^3 + xy^2$$

Taylor series is given by,

$$f(x, y) = f(a, b) + \frac{1}{1!}[(x - a) f_x(a, b) + (y - b) f_y(a - b)]$$
$$+ \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b)$$
$$+ (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x - a)^3 f_{xxx}(a, b) + 3(x - a)^2 (y - b) f_{xyy}(a - b) + 3(x - a)(y - b)^2 f_{xyy}(a, b)$$
$$+ (y - b)^3 f_{yyy}(a - b)] \quad \ldots (1)$$

Here, $a = 1, b = 2$

$f(x, y) = x^3 + y^3 + xy^2$

$\Rightarrow \quad f(1, 2) = (1)^3 + (2)^3 + (1)(2)^2 = 13$

$f_x(x, y) = 3x^2 + y^2$

$\Rightarrow \quad f_x(1, 2) = 3(1)^2 + (2)^2 = 7$

$f_y(x, y) = 3y^2 + 2xy$

$\Rightarrow \quad f_y(1, 2) = 3(2)^2 + 2(1)(2) = 16$

$f_{xy}(x, y) = 2y$

$\Rightarrow \quad f_{xy}(1, 2) = 2(2) = 4$

$f_{xx}(x, y) = 6x$

$\Rightarrow \quad f_{xx}(1, 2) = 6(1) = 6$

$f_{xxx}(x, y) = 6$

$\Rightarrow \quad f_{xxx}(1. 2) = 6$

$f_{xxy}(x, y) = 0$

$\Rightarrow \quad f_{xxy}(1, 2) = 0$

$f_{xyy}(x, y) = 2$

$\Rightarrow \quad f_{xyy}(1, 2) = 2$

$f_{yy}(x, y) = 6y + 2x$

$\Rightarrow \quad f_{yy}(1, 2) = 6(2) + 2(1) = 14$

$f_{yyy}(x, y) = 6$

$\Rightarrow \quad f_{yyy}(1, 2) = 6$

Substituting the corresponding values in equaiton (1),

$$x^3 + y^3 + xy^2 = 13 + \frac{1}{1!}[(x-1)7 + (y-2)16] + \frac{1}{2!}[(x-1)^2\,6 + 2(x-1)(x-2)(4) + (y-2)^2\,14]$$

$$+ \frac{1}{3!}[(x-1)^3\,6 + 3(x-1)^2(y-2)(0) + 3(x-1)(y-2)^2(2) + (y-2)^3(6)]$$

$$= 13 + 7(x-1) + 16(y-2) + \frac{1}{2}[6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2] + \frac{1}{6}[6(x-1)^3$$

$$+ 0 + 6(x-1)(y-2)^2 + 6(y-2)^3]$$

$$= 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2 + (x-1)^3 + (x-1)(y-2)^2 + (y-2)^3$$

$\therefore \; x^3 + y^3 + xy^2 = 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2 + (x-1)^3 + (x-1)(y-2)^2 + (y-2)^3$

Q57. Using Taylor's series expansion, expand e^x siny in powers of x and y as far as terms of the 3rd degree.

OR

Expand e^x sin (y) in powers of x and y upto the third degree terms.

OR

Expand e^xsiny in powers of x and y as far as the terms of the 3rd degree using Taylor's expansion.

Answer :

Let the given function be,

$f(x, y) = e^x \sin y$

$\Rightarrow f(0, 0) = 0$

Consider,

$f_x(x, y) = e^x \sin y \quad ; \quad f_x(0, 0) = 0$

$f_{xy}(x, y) = e^x \cos y \quad ; \quad f_{xy}(0, 0) = 1$

$f_{xx}(x, y) = e^x \sin y \quad ; \quad f_{xx}(0, 0) = 0$

$f_{xyy}(x, y) = -e^x \sin y \; ; \quad f_{xyy}(0, 0) = 0$

$f_{xxx}(x, y) = e^x \sin y \quad ; \quad f_{xxx}(0, 0) = 0$

$f_{xxy}(x, y) = e^x \cos y \quad ; \quad f_{xxy}(0, 0) = 1$

$f_y(x, y) = e^x \cos y \quad ; \quad f_y(0, 0) = 1$

$f_{yy}(x, y) = -e^x \sin y \quad ; \quad f_{yy}(0, 0) = 0$

$f_{yyx}(x, y) = -e^x \sin y \; ; \quad f_{yyx}(0, 0) = 0$

$f_{yyy}(x, y) = -e^x \cos y \quad ; \quad f_{yyy}(x, y) = -1$

From Taylor's series,

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \ldots.$$

$$= 0 + x(0) + y(1) + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)] + \frac{1}{3!}[x^3(0) + 3x^2 y(1) + 3xy^2(0) + y^3(-1)] + \ldots.$$

$$\therefore \quad e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \ldots.$$

Q58. Expand $\tan^{-1}\left(\frac{y}{x}\right)$ as a Taylor series about the point (1, 1) upto 2nd degree terms.

Answer :

Given function is,

$$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$f(1, 1) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$f_x(x, y) = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{\frac{-y}{x^2}}{\frac{x^2 + y^2}{x^2}}$$

$$\Rightarrow \quad f_x(x, y) = \frac{-y}{x^2 + y^2}$$

$$\Rightarrow \quad f_x(1, 1) = \frac{-1}{1+1} = \frac{-1}{2}$$

$$f_y(x, y) = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{1}{x}\frac{1}{\frac{x^2 + y^2}{x^2}} = \frac{x^2}{x(x^2 + y^2)} = \frac{x}{x^2 + y^2}$$

$$\Rightarrow \quad f_y(1, 1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f_{xx}(x, y) = \frac{-y}{(x^2 + y^2)^2}(-2x) = \frac{2xy}{(x^2 + y^2)^2}$$

$$\Rightarrow \quad f_{xx}(1, 1) = \frac{2(1)(1)}{(1+1)^2} = \frac{2}{(2)^2} = \frac{2}{4} = \frac{1}{2}$$

$$f_{xy}(x, y) = \frac{-\left[\left(x^2 + y^2\right)1 - y(2y)\right]}{\left(x^2 + y^2\right)^2}$$

$$= \frac{-\left[x^2 + y^2 - 2y^2\right]}{\left(x^2 + y^2\right)^2}$$

$$f_{xy}(x, y) = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\Rightarrow \quad f_{xy}(1, 1) = \frac{1-1}{(1+1)^2} = 0$$

$$f_{yy}(x, y) = \frac{x}{\left(x^2 + y^2\right)^2}(-2y)$$

$$= \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

$$\Rightarrow \quad f_{yy}(1, 1) = \frac{-2}{(1+1)^2} = \frac{-2}{4} = -\frac{1}{2}$$

From Taylor's series,

$$f(x, y) = f(1,1) + \frac{1}{1!}\left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + (x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{(x-1)}{2} + \frac{(y-1)}{2} + \frac{1}{2}\left[\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2}\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

Q59. Expand sin (xy) in powers of (x – 1) and $\left(y - \frac{\pi}{2}\right)$ up to second degree terms by using Taylor's series.

OR

Expand sin xy at $\left(1.\frac{\pi}{2}\right)$ upto second degree terms using Taylor's series.

Answer :

Given function is,

$$f(x, y) = \sin(xy)$$

$$\Rightarrow \quad f\left(1, \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$f_x(x, y) = y\cos(xy)$$

$$\Rightarrow \quad f_x\left(1, \frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = 0$$

$$f_y(x, y_ = x\cos(xy)$$

$$\Rightarrow \quad f_y\left(1, \frac{\pi}{2}\right) = 1.\cos\frac{\pi}{2} = 0$$

$$f_{xx}(x, y) = -y^2\sin(x, y)$$

$$\Rightarrow \quad f_{xx}\left(1, \frac{\pi}{2}\right) = \frac{-\pi^2}{4} \sin\frac{\pi}{2} = \frac{-\pi^2}{4}$$

$$f_{xy}(x, y) = -xy \sin(x, y)$$

$$\Rightarrow \quad f_{xy}\left(1, \frac{\pi}{2}\right) = \frac{-\pi}{2} \sin\frac{\pi}{2} = \frac{-\pi}{2}$$

$$f_{yy}(x, y) = -x^2 \sin(x, y)$$

$$\Rightarrow \quad f_{yy}\left(1, \frac{\pi}{2}\right) = -1 \sin\frac{\pi}{2} = -1$$

From Taylor's series,

$$f(x, y) = f\left(1, \frac{\pi}{2}\right) + (x-1) f_x\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right) f_y\left(1, \frac{\pi}{2}\right)$$

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1, \frac{\pi}{2}\right) + 2(x-1)\left(y - \frac{\pi}{2}\right) f_{xy}\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\left(1, \frac{\pi}{2}\right)\right]$$

Substituting the corresponding values in the above equation,

$$\sin(xy) = 1 + (x-1)(0) + \left(y - \frac{\pi}{2}\right)(0) + \frac{1}{2!}\left[(x-1)^2\left(\frac{-\pi^2}{4}\right) + 2(x-1)\left(y - \frac{\pi}{2}\right)\left(\frac{-\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 (-1)\right]$$

$$= 1 + \frac{1}{2}\left[(x-1)^2\left(\frac{-\pi}{4}\right)^2 - 2(x-1)\left(y - \frac{\pi}{2}\right)\frac{\pi}{2} - \left(y - \frac{\pi}{2}\right)^2\right]$$

$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2$$

$$\therefore \quad \sin(xy) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2.$$

3.8 MAXIMUM AND MINIMUM VALUES OF FUNCTIONS OF TWO VARIABLES, LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Q60. When a function f(x, y), with usual notations of partial differential coefficients, will have maximum, minimum and can't be decided?

Answer :

Given function is,

$f(x, y)$

Let (a, b) be the point at which maximum or minimum is determined. The condition for the function $f(x, y)$ to have maximum or minimum at a point (a, b) are $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Let,

$$\frac{\partial^2}{\partial x^2} f(a,b) = r$$

$$\frac{\partial^2}{\partial x \partial y} f(a,b) = t$$

$$\frac{\partial^2}{\partial y^2} f(a,b) = s$$

Then,

Case (i)

Function $f(a, b)$ is said to have maximum value if $rt - s^2 > 0$ and $r < 0$

Case (ii)

Function $f(a, b)$ is said to have minimum value if $rt - s^2 > 0$ and $r > 0$

Case (iii)

Function $f(a, b)$ does not have an extreme value i.e, neither maximum or minimum if $rt - s^2 < 0$.

Case (iv)

Function $f(a, b)$ fails to have maximum or minimum value and can't be decided if $rt - s^2 = 0$.

Q61. Explain Lagrange's method of undetermined multipliers.

Answer :

Let u be a function of three variables x, y, z given as,

$$u = f(x, y, z) \quad \ldots (1)$$

Let the variables are connected by a relation,

$$\phi(x, y, z) = 0 \quad \ldots (2)$$

Since, u is to have stationary value,

$$\frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du = 0 \quad \ldots (3)$$

Similarly equation (2) becomes,

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi = 0 \quad \ldots (4)$$

Multiplying equation (4) by λ,

$$\lambda\left(\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz\right) = 0$$

$$\lambda\frac{\partial \phi}{\partial x}dx + \lambda\frac{\partial \phi}{\partial y} + \lambda\frac{\partial \phi}{\partial z}dz = 0 \quad \ldots (5)$$

Where

λ – Lagrange's multiplier

Adding equations (3) and (5),

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \lambda\frac{\partial \phi}{\partial x}dx + \lambda\frac{\partial \phi}{\partial y}dy + \lambda\frac{\partial \phi}{\partial z}dz = 0$$

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad \ldots (6)$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad \ldots (7)$$

$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad \ldots (8)$$

Equations (2), (6) (7) and (8) give the values of x, y, z and λ for which u is stationary.

Q62. Write the steps involved in Lagrange's multiplier method.

Ans :

Let, $u = f(x, y, z)$ be a function of three variables connected by a relation $\phi(x, y, z) = 0$

Step 1

The function is represented as,

$$F = u + \lambda\,\phi(x, y, z)$$

$$\Rightarrow \quad F = \phi(x, y, z) + \lambda\,\phi(x, y, z) \quad \ldots (1)$$

Where, λ – Lagrange's multiplier

Step 2

Differentiate equation (1) with respect to x, y and z and equate them to zero

i.e., $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Step 3

Solve the above three equations along with the relation $\phi(x, y, z)$ to obtain the values of x, y and z.

Step 4

Substitute the x, y, z values in the function u to obtain stationary value of u.

Q63. Examine the function $f(x, y) = x^3y^2(12 - x - y)$ for extreme values.

Answer :

Given function is,

$$f(x, y) = x^3y^2(12 - x - y)$$

$$\Rightarrow \quad f = f(x, y) = 12x^3y^2 - x^4y^2 - x^3y^3 \qquad \text{... (1)}$$

Differentiating equation (1) partially with respect to 'x',

$$\frac{\partial f}{\partial x} = 36x^2y^2 - 4x^3y^2 - 3x^2y^3 \qquad \text{... (2)}$$

Differentiating equation (1) partially with respect to 'y',

$$\frac{\partial f}{\partial y} = 24x^3y - 2x^4y - 3x^3y^2 \qquad \text{... (3)}$$

Differentiating equation (2) partially with respect to 'x',

$$\frac{\partial^2 f}{\partial x^2} = 72xy^2 - 12x^2y^2 - 6xy^3 = r$$

Differentiating equation (2) partially with respect to 'y',

$$\frac{\partial^2 f}{\partial x \partial y} = 72x^2y - 8x^3y - 9x^2y^2 = s$$

Differentiating equation (3) partially with respect to 'y',

$$\frac{\partial^2 f}{\partial y^2} = 24x^3 - 2x^4 - 6x^3y = t$$

The conditions for maximum or minimum value i.e., extreme value are,

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0$$

From equation (2),

$$\frac{\partial f}{\partial x} = 0$$

$$\Rightarrow \quad 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$\Rightarrow \quad x^2y^2[36 - 4x - 3y] = 0$$

$$\Rightarrow \quad x = 0 \text{ or } y = 0$$

$$\text{or } 4x + 3y = 36$$

From equation (3),

$$\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \quad 24x^3y - 2x^4y - 3x^3y^2 = 0$$

$$\Rightarrow \quad x^3y[24 - 2x - 3y] = 0$$

$$\Rightarrow \quad x = 0 \text{ or } y = 0 \text{ or } 2x + 3y = 24$$

$\Rightarrow$ $(x, y) = (0, 0)$

$\Rightarrow$ $4x + 3y = 36$

$2x + 3y = 24$

$- \quad - \quad -$

$2x = 12$

$\Rightarrow$ $x = 6$

$2(6) + 3y = 24$

$\Rightarrow$ $3y = 12$

$\Rightarrow$ $y = 4$

$\Rightarrow$ $(x, y) = (6, 4)$

$\therefore$ The stationary points are (0, 0), (6, 4)

Consider,

$$rt - s^2 = (72xy^2 - 12x^2y^2 - 6xy^3)(24x^3 - 2x^4 - 6x^3y) - (72x^2y - 8x^3y - 9x^2y^2)^2$$

$$rt - s^2|_{(0,0)} = (72(0)(0)^2 - 12(0)^2(0)^2 - 6(0)(0)^3)(24(0)^3 - 2(0)^4) - [72(0)^3(0) - 6(0)^2(0) - 8(0)^3(0) - 9(0)^2(0)^2]^2$$

$$= 0$$

$\therefore$ f does not have maximum or minimum value at (0, 0)

$$rt - s^2|_{(6,4)} = [72(6)(4)^2 - 12(6)^2(4)^2 - 6(6)(4)^3] \times [24(6)^3 - 2(6)^4 - 6(6)^3(4)] - [(72(6)^2(4) - 8(6)^3(4) - 9(6)^2(4)^2)]^2$$

$$= (6912 - 6912 - 2304)(5184 - 2592 - 5184) - [10368 - 6912 - 5184]^2$$

$$= 5971968 - (1728)^2$$

$$= 5971968 - 2985984$$

$$= 2985984 > 0$$

$\therefore$ f has maximum value at (6, 4)

Maximum value is obtained by substituting $x = 6$, $y = 4$ in equation (1)

$$f(x, y) = f(6, 4) = 12(6)^3(4)^2 - (6)^4(4)^2 - (6)^3(4)^3$$

$$= 41472 - 20736 - 13824$$

$$= 6912$$

$\therefore$ Maximum value = 6912.

Q64. Divide 24 into three points such that the continued product of the first, square of the second and cube of the third is maximum.

Answer :

Let x, y and z be three points.

$\Rightarrow$ $x + y + z = 24$ ... (1)

And the product of the first, square of the second and cube of the third point is maximum i.e., to maximize $(x \times y^2 \times z^3)$

Let, $\phi = xy^2z^3$ ... (2)

From equation (1),

$x = 24 - y - z$ ... (3)

Substituting the value of x from equation (3) in equation (2),

$$\phi = (24 - y - z)\, y^2 z^3$$

$$\Rightarrow \quad \phi(y, z) = 24y^2z^3 - y^3z^3 - y^2z^4 \qquad \ldots (4)$$

Differentiating equation (4) partially with respect to 'y',

$$\frac{\partial \phi}{\partial y} = 48yz^3 - 3y^2z^3 - 2yz^4 \qquad \ldots (5)$$

Differentiating equation (4) partially with respect to 'z',

$$\frac{\partial \phi}{\partial z} = 72y^2z^2 - 3y^3z^2 - 4y^2z^3 \qquad \ldots (6)$$

For $\phi(y, z)$ to be maximum,

$$\frac{\partial \phi}{\partial y} = 0 \text{ and } \frac{\partial \phi}{\partial z} = 0$$

$\therefore$ Equation (5) becomes,

$$\frac{\partial \phi}{\partial y} = 48yz^3 - 3y^2z^3 - 2yz^4 = 0$$

$$\Rightarrow yz^3[48 - 3y - 2z] = 0$$

Since, $yz \neq 0$,

$$48 - 3y - 2z = 0$$

$$\Rightarrow \quad -3y - 2z = -48$$

$$\Rightarrow \quad 3y + 2z = 48 \qquad \ldots (7)$$

Equation (6) becomes,

$$\frac{\partial \phi}{\partial z} = 72y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0$$

$$\Rightarrow \quad y^2z^2[72 - 3y - 4z] = 0$$

Since, $yz \neq 0$

$$72 - 3y - 4z = 0$$

$$\Rightarrow \quad -3y - 4z = -72$$

$$\Rightarrow \quad 3y + 4z = 72 \qquad \ldots (8)$$

Solving equations (7) and (8),

$$y = 8 \text{ and } z = 12$$

Substituting the values of y and z in equation (1),

$$x + y + z = 24$$

$$\Rightarrow \quad x = 24 - y - z$$

$$\Rightarrow \quad x = 24 - 8 - 12$$

$$\Rightarrow \quad x = 24 - 20$$

$$\Rightarrow \quad x = 4$$

$\therefore \quad x = 4, y = 8$ and $z = 12$

$$r = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}[48yz^3 - 3y^2z^3 - 2yz^4]$$

$\therefore \quad r = 48z^3 - 6yz^3 - 2z^4 \qquad \ldots (9)$

$$s = \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial z}\right]$$

$$= \frac{\partial}{\partial y}[72y^2z^2 - 3y^3z^2 - 4y^2z^3]$$

$\therefore \quad s = 144\, yz^2 - 9y^2z^2 - 8yz^3 \qquad \ldots (10)$

$$t = \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial \phi}{\partial z}\right]$$

$$= \frac{\partial}{\partial z}[72y^2z^2 - 3y^3z^2 - 4y^2z^3]$$

$\therefore \quad t = 144\, y^2z - 6y^3z - 12y^2z^2 \qquad \ldots (11)$

At $y = 8$ and $z = 12$

Equation (9) becomes,

$$r = 48[12]^3 - 6[8][12]^3 - 2[12]^4$$

$$\Rightarrow r = 82944 - 82944 - 41472$$

$$\Rightarrow r = -41472 \qquad \ldots (12)$$

Equation (10) becomes,

$$s = 144[8]\,[12]^2 - 9[8]^2\,[12]^2 - 8[8]\,[12]^3$$

$$\Rightarrow s = 165888 - 82944 - 110592$$

$$\Rightarrow s = -27648 \qquad \ldots (13)$$

Equation (11) becomes,

$$t = 144[8]^2\,[12] - 6[8]^3\,[12] - 12[8]^2\,[12]^2$$

$$\Rightarrow t = 110592 - 36864 - 110592$$

$$\Rightarrow t = -36864 \qquad \ldots (14)$$

Consider,

$$rt - s^2 = [-41472 \times -36864] - [-27648]^2$$

$$= 764.41 \times 10^6 > 0$$

$\therefore rt - s^2 > 0$ and $r < 0$

$\therefore \phi = xy^2z^3$ is maximum when 24 is divided into $x = 4$, $y = 8$ and $z = 12$.

Q65. Find the stationary points of u(x, y) = sinx siny sin(x + y) where $0 < x < \pi$, $0 < y < \pi$ and find the maximum u.

Answer :

Given function is,

$$u(x, y) = \sin x \sin y \sin(x + y)$$

Where, $0 < x < \pi$,

$$0 < y < \pi \qquad \ldots (1)$$

$$u = \sin x \sin y \sin(x + y)$$

Differentiating equation (1) partially with respect to x,

$$\frac{\partial u}{\partial x} = \cos x \sin y \sin(x+y) + \sin x \cos(x+y) \sin y$$
$$= \sin y [\cos x \sin(x+y) + \sin x \cos(x+y)]$$

$$\therefore \frac{\partial u}{\partial x} = \sin y \sin(2x+y)$$

$[\because \sin(A+B) = \cos A \sin B + \sin A \cos B]$

For stationary point,

$$\frac{\partial u}{\partial x} = 0$$

$\Rightarrow \sin y \sin(2x+y) = 0$

$\Rightarrow \sin y = 0$ or $\sin(2x+y) = 0$

$[\because 0 < x < \pi, 0 < y < \pi]$

$\Rightarrow y = 0$ or $2x + y = \pi$ ... (2)

Differentiating equation (1) partially with respect to y,

$$\frac{\partial u}{\partial y} = \sin x \sin y \cos(x+y) + \sin x \cos y \sin(x+y)$$
$$= \sin x [\sin y \cos(x+y) + \sin(x+y) \cos y]$$

$$\therefore \frac{\partial u}{\partial y} = \sin x \sin(x+2y)$$

For stationary point,

$$\frac{\partial u}{\partial y} = 0$$

$\Rightarrow \sin x \sin(x+2y) = 0$

$\Rightarrow \sin x = 0$ or $\sin(x+2y) = 0$

$\Rightarrow x = 0$ or $x + 2y = \pi$ ... (3)

Solving equations (2) and (3),

$\Rightarrow x = \frac{\pi}{3}, \ y = \frac{\pi}{3}$

$\therefore$ Stationary point is $x = \frac{\pi}{3}, \ y = \frac{\pi}{3}$

$$r = \frac{\partial^2 u}{\partial x^2} = 2\sin y \cos(2x+y)$$

$$= 2\sin\frac{\pi}{3} \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$$

$$\therefore r = -\frac{2\sqrt{3}}{2} = -\sqrt{3}$$

$$s = \frac{\partial^2 u}{\partial x \partial y}$$

$$= \sin x \cos(x+2y) + \cos x \sin(x+2y)$$

$$= \sin(2x+2y)$$

$$= \sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right)$$

$$\therefore s = -\frac{\sqrt{3}}{2} \qquad \left[\because x = y = \frac{\pi}{3}\right]$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2 \sin x \cos(x+2y)$$

$$\therefore t = 2\sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) - \frac{2\sqrt{3}}{2} = -\sqrt{3}$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$$

$$\therefore rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$\therefore u$ will be maximum at $x = \frac{\pi}{3}, y = \frac{\pi}{3}$

$$\text{Maximum value} = \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} \ \frac{\sqrt{3}}{2} \ \sin\left(\frac{2\pi}{3}\right)$$

$$= \frac{3}{4} \ \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

$\therefore$ Maximum value $= \frac{3\sqrt{3}}{8}$

$\therefore$ Stationary point $= \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Q66. Find the absolute maximum and minimum values for the function $f(x, y) = x^2 - y^2 - 2y$ in the closed region $R : x^2 + y^2 \leq 1$.

Answer : **Dec.-16, Q14(b)**

Given that,

$$f(x, y) = x^2 - y^2 - 2y \quad \text{... (1)}$$

$$R : x^2 + y^2 \leq 1 \quad \text{... (2)}$$

Differentiating equation (1) partially with respect to x

$$\frac{\partial f}{\partial x} = 2x \quad \text{... (3)}$$

Equating the value of $\frac{\partial f}{\partial x}$ to zero in order to determine the extreme point

$$\frac{\partial f}{\partial x} = 2x = 0$$

$\Rightarrow 2x = 0$

$\Rightarrow x = 0$

Differentiating equation (2) partially with respect to y

$$\frac{\partial f}{\partial y} = -2y - 2 \quad \text{... (4)}$$

Equating the value of $\frac{\partial f}{\partial y}$ to zero in order to determine the extreme point

$$\frac{\partial f}{\partial y} = -2y - 2 = 0$$

$\Rightarrow \quad -2y - 2 = 0$

$\Rightarrow \quad -2y = 2$

$\Rightarrow \quad y = -1$

$\therefore$ The critical point $(x, y) = (0, -1)$

Differentiating equation (3) partially with respect to x

$$p = \frac{\partial^2 f}{\partial x^2} = 2$$

Differentiating equation (3) partially with respect to y

$$q = \frac{\partial^2 f}{\partial x \partial y} = 0$$

Differentiating equation (4) partially with respect to y

$$r = \frac{\partial^2 f}{\partial y^2} = -2$$

$$pr - q^2 = 2(-2) - 0^2$$

$$= -4 < 0$$

As $pr - q^2 < 0$, the function $f(x, y)$ does not have extreme value.

As $\frac{\partial^2 f}{\partial x^2} > 0$, the function has relative minimum value.

The relative minimum value at $(0, -1)$ is given as,

$$f(0, -1) = 0^2 - (-1)^2 - 2(-1)$$

$$= 0 - 1 + 2 = 1$$

From equations (2);

$$x^2 = 1 - y^2 \quad \text{... (5)}$$

Substituting equation (5) in equation (1)

$\Rightarrow \quad g(y) = (1 - y^2) - y^2 - 2y$

$\Rightarrow \quad g(y) = 1 - 2y^2 - 2y \quad \text{... (6)}$

Differentiating equation (6) partially with respect to y

$$\frac{\partial g}{\partial y} = 0 - 4y - 2$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = -4y - 2 \quad \text{... (7)}$$

Equating $\frac{\partial g}{\partial y}$ to zero;

$\Rightarrow \quad -4y - 2 = 0$

$\Rightarrow \quad -4y = 2$

$$\Rightarrow \quad y = \frac{-1}{2}$$

Substituting $y = \frac{-1}{2}$ in equation (5)

$$x^2 = 1 - \left(\frac{-1}{4}\right)^2$$

$$\Rightarrow \quad x^2 = 1 - \frac{1}{4}$$

$$\Rightarrow \quad x^2 = \frac{3}{4}$$

$$\Rightarrow \quad x = \pm \frac{\sqrt{3}}{2}$$

$\therefore$ The critical region on the boudary is $(x, y) = \left(\pm \frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

Differentiating equation (7) partially with respect to y

$$\frac{\partial^2 g}{\partial y^2} = -4 < 0$$

As $\frac{\partial^2 g}{\partial y^2} < 0$, the function $f(x, y)$ has relative maximum value.

$$f\left(\pm \frac{\sqrt{3}}{2}, \frac{-1}{2}\right) = \left(\pm \frac{\sqrt{3}}{2}\right)^2 - \left(\frac{-1}{2}\right)^2 - 2\left(\frac{-1}{2}\right)$$

$$= \frac{3}{4} - \frac{1}{4} + 1 = \frac{3}{2}$$

$\therefore$ The given function has relative minimum value as 1 at $(0, -1)$ and relative maximum value as $\frac{3}{2}$ at $\left(\pm \frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

Q67. Find the maximum value of $x^2y^3z^4$ subject to the condition $2x + 3y + 4z = a$.

Answer :

Consider the given functions as,

$$\phi(x, y, z) = 2x + 3y + 4z - a = 0 \quad \text{... (1)}$$

And $u(x, y, z) = x^2y^3z^4 \quad \text{... (2)}$

According to Lagrange's function,

$$F(x, y, z) = u(x, y, z) + \lambda\phi(x, y, z) \quad \text{... (3)}$$

Where, λ = Lagrange's multiplier.

Substituting equations (1) and (2) in equation (3),

$$F(x, y, z) = x^2y^3z^4 + \lambda(2x + 3y + 4z - a) \quad \text{... (4)}$$

Differentiating equation (4) partially with respect to 'x',

$$\frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda$$

Differentiating equation (4) partially with respect to 'y',

$$\frac{\partial F}{\partial y} = 3y^2x^2z^4 + 3\lambda$$

Differentiating equation (4) partially with respect to 'z',

$$\frac{\partial F}{\partial z} = 4z^3x^2y^3 + 4\lambda$$

For maximum or minimum value, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

Consider,

$$\frac{\partial F}{\partial x} = 0$$

$\Rightarrow \quad 2xy^3z^4 + 2\lambda = 0$

$\Rightarrow \quad 2(xy^3z^4 + \lambda) = 0$

$\Rightarrow \quad xy^3z^4 + \lambda = 0$

Multiplying and dividing by 'x',

$$\Rightarrow \frac{x}{x}.xy^3z^4 + \lambda = 0$$

$$\Rightarrow \frac{1}{x}.x^2y^3z^4 = -\lambda$$

$$\Rightarrow \frac{u}{x} = -\lambda \qquad [\because \text{From equation (2)}]$$

$$\therefore x = \frac{-u}{\lambda}$$

$$\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow 3y^2x^2z^4 + 3\lambda = 0$$

$$\Rightarrow 3(y^2x^2z^4 + \lambda) = 0$$

$$\Rightarrow y^2x^2z^4 + \lambda = 0$$

Multiplying and dividing by 'y',

$$\Rightarrow \frac{y}{y}.y^2x^2z^4 + \lambda = 0$$

$$\Rightarrow \frac{y^3x^2z^4}{y} = -\lambda$$

$$\Rightarrow \frac{u}{y} = -\lambda \qquad [\because \text{From equation (2)}]$$

$$\therefore y = \frac{-u}{\lambda}$$

$$\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow 4z^3x^2y^3 + 4\lambda = 0$$

$$\Rightarrow 4(z^3x^2y^3 + \lambda) = 0$$

$$\Rightarrow z^3x^2y^3 + \lambda = 0$$

Multiplying and dividing by 'z',

$$\Rightarrow \frac{z}{z}.z^3x^2y^3 + \lambda = 0$$

$$\Rightarrow \frac{1}{z}.x^2y^3z^4 = -\lambda$$

$$\Rightarrow \frac{u}{z} = -\lambda \qquad [\because \text{From equation (2)}]$$

$$\therefore z = \frac{-u}{\lambda}$$

$$x = y = z = \frac{-u}{\lambda} \qquad \ldots (5)$$

Substituting the values of x, y and z in equation (1),

$$\phi(x, y, z) = 2x + 3y + 4z - a = 0$$

$$\Rightarrow 2\left(\frac{-u}{\lambda}\right) + 3\left(\frac{-u}{\lambda}\right) + 4\left(\frac{-u}{\lambda}\right) - a = 0$$

$$\Rightarrow \frac{-u}{\lambda}[2 + 3 + 4] = a$$

$$\Rightarrow \frac{-u}{\lambda}[9] = a$$

$$\Rightarrow -9u = a\lambda$$

$$\therefore \lambda = \frac{-9u}{a}$$

Substituting the value of 'λ' in equation (5),

$$\Rightarrow x = y = z = \frac{-u}{\lambda}$$

$$= \frac{-u}{\left(\frac{-9u}{a}\right)}$$

$$= \frac{-au}{-9u} = \frac{a}{9}$$

$$\therefore x = y = z = \frac{a}{9}$$

The maximum value is given as,

$$= \left(\frac{a}{9}\right)^2\left(\frac{a}{9}\right)^3\left(\frac{a}{9}\right)^4 = \left(\frac{a}{9}\right)^{2+3+4} = \left(\frac{a}{9}\right)^9$$

$\therefore$ The maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$.

Q68. Find the maximum and minimum values of $x + y + z$ subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Answer :

Given function is,

$$f(x, y, z) = x + y + z$$

It is basically a constrained extremum problem, where a function 'f' is subjected to the constraint.

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

Substituting both the function and its constraint in the following auxiliary function.

$$F(x, y, z) = x + y + z + \lambda\left[\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right] \qquad \ldots (1)$$

Differentiating equation (1) partially with respect to x, y, z and then equating the results to zero.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x+y+z) + \lambda \frac{\partial}{\partial x}\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right]$$

$$= 1 + \left[\frac{-\lambda}{x^2}\right]$$

$$= 1 - \frac{\lambda}{x^2}$$

$$= \frac{x^2-\lambda}{x^2}$$

$$\therefore \quad \frac{\partial F}{\partial x} = x^2 - \lambda \qquad \text{... (2)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x+y+z) + \lambda \frac{\partial}{\partial y}\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right]$$

$$= [1] + \left[\frac{-\lambda}{y^2}\right]$$

$$= \frac{y^2-\lambda}{y^2}$$

$$\therefore \quad \frac{\partial F}{\partial y} = y^2 - \lambda \qquad \text{... (3)}$$

Similarly,

$$\frac{\partial F}{\partial z} = 1 - \frac{\lambda}{z^2} = \frac{z^2-\lambda}{z^2}$$

$$\therefore \quad \frac{\partial F}{\partial z} = \frac{z^2-\lambda}{z^2} \qquad \text{... (4)}$$

Equating equations (2), (3) and (4) to zero,

$$x^2 - \lambda = 0, \quad y^2 - \lambda = 0, \quad z^2 - \lambda = 0$$

$$x^2 = \lambda, \quad y^2 = \lambda, \quad z^2 = \lambda$$

$$x = \pm\sqrt{\lambda}, \quad y = \pm\sqrt{\lambda} \text{ and } z = \pm\sqrt{\lambda}$$

Substituting the values of x, y, z in the given constraint,

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}} = 1$$

$$\Rightarrow \quad \frac{1+1+1}{\sqrt{\lambda}} = 1$$

$$\Rightarrow \quad \frac{3}{\sqrt{\lambda}} = 1$$

$$\Rightarrow \quad 3 = \sqrt{\lambda}$$

Squaring on both sides,

$$\Rightarrow \quad \lambda = 3^2 = 9$$

$$\therefore \quad \lambda = 9$$

Substituting the value of λ in x, y and z,

$$x = \pm 3, \; y = \pm 3, \; z = \pm 3$$

$\therefore$ The extreme i.e., maximum and minimum values are + 9 and – 9.

Q69. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Answer :

Let, the coordinates of the rectangular parallelopiped be (x, y, z).

The volume of the rectangular parallelopiped,

$$V = 8xyz \qquad \text{... (1)}$$

Maximum volume is to be calculated such that it is inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Let,

$$F = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \qquad \text{... (2)}$$

The condition for extreme value is given by,

$$dV = yz\,dx + zx\,dy + xy\,dz = 0 \qquad \text{... (3)}$$

Differentiating equation (2),

$$dF = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0$$

$$\Rightarrow dF = \frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \qquad \text{... (4)}$$

Multiplying equation (4) by λ and adding equations (3) and (4),

$$(yz\,dx + zx\,dy + xy\,dz) + \left(\frac{\lambda x}{a^2}dx + \frac{\lambda y}{b^2}dy + \frac{\lambda z}{c^2}dz\right) = 0$$

$$\left(yz + \frac{\lambda x}{a^2}\right)dx + \left(xz + \frac{\lambda y}{b^2}\right)dy + \left(xy + \frac{\lambda z}{c^2}\right)dz = 0$$

Equating the coefficients of dx, dy, dz to zero,

$$\Rightarrow \quad \left(yz + \frac{\lambda x}{a^2}\right) = 0 \qquad \text{... (5)}$$

$$\Rightarrow \quad \left(xz + \frac{\lambda y}{b^2}\right) = 0 \qquad \text{... (6)}$$

$$\Rightarrow \quad \left(xy + \frac{\lambda z}{c^2}\right) = 0 \qquad \text{... (7)}$$

From equations (5), (6) and (7),

$$\Rightarrow \quad \lambda = \frac{-yza^2}{x} = \frac{-zxb^2}{y} = \frac{-xyc^2}{z}$$

$$\Rightarrow \quad \frac{a^2yz}{x} = \frac{b^2zx}{y} = \frac{c^2xy}{z} \qquad \text{... (8)}$$

Dividing throughout equation (8) by xyz,

$$\Rightarrow \quad \frac{a^2yz}{x^2yz} = \frac{b^2zx}{xy^2z} = \frac{c^2xy}{xyz^2}$$

$$\Rightarrow \quad \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Substituting, $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$ in equation (2),

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \qquad \left[\because \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{x^2}{a^2}\right]$$

$$\Rightarrow \quad \frac{3x^2}{a^2} = 1$$

$$\Rightarrow \quad x^2 = \frac{a^2}{3}$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{3}}$$

Similarly, $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$.

Thus, the stationary points are $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

Differentiating equation (2) partially with respect to 'x' and keeping y as constant,

$$\frac{1}{a^2}(2x) + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = \frac{-2x}{a^2}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-2x}{a^2} \times \frac{c^2}{2z}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-xc^2}{a^2z} \qquad \text{... (9)}$$

Similarly, differentiating equation (1) partially with respect to 'x' and keeping y as constant,

$$\frac{\partial V}{\partial x} = 8yz + 8xy\frac{\partial z}{\partial x} \qquad \text{... (10)}$$

Substituting equation (9) in equation (10),

$$= 8yz + 8xy\left(\frac{-c^2x}{a^2z}\right)$$

$$\therefore \quad \frac{\partial V}{\partial x} = 8yz - \frac{8x^2yc^2}{a^2z}$$

Again differentiating equation (5) partially with respect to 'x' keeping 'y' as constant,

$$\frac{\partial^2 V}{\partial x^2} = 8y\left(\frac{\partial z}{\partial x}\right) - \frac{8yc^2}{a^2}\frac{\partial}{\partial x}\left(\frac{x^2}{z}\right)$$

$$\therefore \quad \frac{\partial^2 V}{dx^2} = 8y\left(\frac{-c^2x}{a^2z}\right) - \frac{16c^2xy}{a^2z} - \frac{8c^2x^2y}{a^2z}\cdot\frac{c^2x}{a^2z}$$

Since, the value of $\frac{\partial^2 V}{\partial x^2}$ is negative,

The maximum value is obtained at the stationary point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

From equation (1), $V = 8xyz$

$$\therefore \text{ Maximum value, } V_{max} = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}}$$

$$V_{max} = \frac{8abc}{3\sqrt{3}}.$$

Q70. Find the shortest and longest distances from the point (1, 2, – 1) to the sphere $x^2 + y^2 + z^2 = 24$.

Answer :

Given equation of sphere is,

$$x^2 + y^2 + z^2 = 24 \qquad \ldots (1)$$

Point (1, 2, – 1)

Let (x, y, z) be the required point on the sphere

Let d be the distance between the points (x, y, z) and (1, 2, – 1)

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

$$\Rightarrow \quad d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$$

Let $f = (x-1)^2 + (y-2)^2 + (z+1)^2$

$$g = x^2 + y^2 + z^2 - 24$$

The auxiliary function is given by

$$F = f + \lambda g$$

Where λ is Lagrange's multiplier

i.e., $F = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda\,(x^2 + y^2 + z^2 - 24)$

The necessary conditions for maximum or minimum values are,

$$\frac{\partial F}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial y} = 0$$

i.e., $2(x-1) + 2\lambda x = 0$

$$\Rightarrow \quad (x-1) = -\lambda x$$

$$\Rightarrow \quad \frac{x-1}{x} = -\lambda$$

$$\Rightarrow \quad \frac{-x+1}{x} = \lambda$$

$$\Rightarrow \quad -1+\frac{1}{x}=\lambda$$

$$\Rightarrow \quad \frac{1}{x}=\lambda+1$$

$$\Rightarrow \quad x=\frac{1}{\lambda+1} \qquad \ldots (2)$$

$$\frac{\partial F}{\partial y}=0$$

i.e., $2(y-2)+2\lambda y=0$

$$\Rightarrow \quad (y-2)=-\lambda y$$

$$\Rightarrow \quad \frac{y-2}{y}=-\lambda$$

$$\Rightarrow \quad \frac{-y+2}{y}=\lambda$$

$$\Rightarrow \quad -1+\frac{2}{y}=\lambda$$

$$\Rightarrow \quad \frac{2}{y}=\lambda+1$$

$$\Rightarrow \quad y=\frac{2}{\lambda+1} \qquad \ldots (3)$$

$$\frac{\partial F}{\partial z}=0$$

i.e., $2(z+1)+2\lambda z=0$

$$\Rightarrow \quad z+1=-\lambda z$$

$$\Rightarrow \quad \frac{z+1}{z}=-\lambda$$

$$\Rightarrow \quad -1-\frac{1}{z}=\lambda$$

$$\Rightarrow \quad \frac{-1}{z}=\lambda+1$$

$$\Rightarrow \quad z=\frac{-1}{\lambda+1} \qquad \ldots (4)$$

Substituting the values of x, y and z in equation (1),

$$\Rightarrow \quad \left(\frac{1}{\lambda+1}\right)^2+\left(\frac{2}{\lambda+1}\right)^2+\left(\frac{-1}{\lambda+1}\right)^2=24$$

$$\Rightarrow \quad \frac{1}{(\lambda+1)^2}+\frac{4}{(\lambda+1)^2}+\frac{1}{(\lambda+1)^2}=24$$

$$\Rightarrow \quad \frac{6}{(\lambda+1)^2}=24$$

$$\Rightarrow \quad (\lambda+1)^2=\frac{6}{24}$$

$$\Rightarrow \quad (\lambda+1)^2=\frac{1}{4}$$

$$\Rightarrow \quad (\lambda+1)=\pm\frac{1}{2}$$

$$\lambda+1=\frac{1}{2} \quad ; \quad \lambda+1=\frac{-1}{2}$$

$$\lambda=\frac{1}{2}-1 \quad ; \quad \lambda=\frac{-1}{2}-1$$

$$\lambda=\frac{-1}{2} \quad ; \quad \lambda=\frac{-3}{2}$$

Substituting $\lambda=\frac{-1}{2}$ in equations (2), (3) and (4)

$$x=\frac{1}{\lambda+1}=\frac{1}{-\frac{1}{2}+1}=\frac{1}{\frac{1}{2}}=2$$

$$y=\frac{2}{\lambda+1}=\frac{2}{\frac{-1}{2}+1}=\frac{2}{\frac{1}{2}}=4$$

$$z=\frac{-1}{\lambda+1}=\frac{-1}{\frac{-1}{2}+1}=\frac{-1}{\frac{1}{2}}=-2$$

$\therefore \quad (x, y, z)=(2, 4, -2)$

Substituting $\lambda=\frac{-3}{2}$ in equations 2, 3, and 4

$$x=\frac{1}{\lambda+1}=\frac{1}{\frac{-3}{2}+1}=\frac{1}{\frac{-1}{2}}=-2$$

$$y=\frac{2}{\lambda+1}=\frac{2}{\frac{-3}{2}+1}=\frac{2}{\frac{-1}{2}}=-4$$

$$z=\frac{-1}{\lambda+1}=\frac{-1}{\frac{-3}{2}+1}=\frac{-1}{\frac{-1}{2}}=2$$

$\therefore \quad (x, y, z)=(2, 4, -2)$

At the point (2, 4, – 2)

The distance d at $(2, 4, -2)$ is,

$$d=\sqrt{(2-1)^2+(4-2)^2+(-2+1)^2}$$

$$=\sqrt{(1)^2+(2)^2+(-1)^2}$$

$$=\sqrt{1+4+1}$$

$\therefore \quad d=\sqrt{6}$ units ... (5)

At the point (– 2, – 4, 2)

The distance d at $(-2, -4, 2)$ is,

$$d=\sqrt{(-2-1)^2+(-4-2)^2+(2+1)^2}$$

$$=\sqrt{(-3)^2+(-6)^2+(3)^2}$$

$$=\sqrt{9+36+9}$$

$$=\sqrt{54}$$

$\therefore \quad d=3\sqrt{6}$ units ... (6)

It can be seen from equations (5) and (6) that shortest distance is $\sqrt{6}$ units and longest distance is, $3\sqrt{6}$ units

$\therefore$ Shortest distance = $\sqrt{6}$ units

Longest distance = $3\sqrt{6}$ units.