UNIT 1

SEQUENCES AND SERIES

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. Define sequence with an example.

Answer :

A sequence is defined as an ordered set of real numbers such as, $u_1, u_2, u_3 \ldots u_n$.

Basically, a sequence is represented as "u_n".

Example

1, 3, 5, 7 .... $(2n - 1)$.

Q2. Define limit.

Answer :

A sequence tends to a limit 'l' if for every positive number $(\varepsilon > 0)$, a value N of n can be obtained.

Such that,

$|u_n - l| < \varepsilon \; \forall \; n \geq N$

A limit can be represented as,

$$\lim_{n \to \infty} (u_n) = l$$

(or)

$$(u_n) \to l \text{ as } n \to \infty$$

Q3. Define convergent sequence and give an example.

Answer :

A sequence is said to be convergent sequence if it has finite limit.

Example

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots \frac{1}{n^2}.$$

Q4. Define divergent sequence with an example.

Answer :

Divergent Sequence

A sequence which is not convergent is called as divergent sequence.

(or)

Let $\{s_n\}$ be a sequence. Then $\{s_n\}$ is said to be divergent sequence if $\lim\limits_{n \to \infty} s_n = \pm\infty$ (or) $-\infty$.

Example

Let $\{s_n\} = \{n^2\}$

$$\Rightarrow \quad s_n = n^2$$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} n^2$$

$$= +\infty$$

$\therefore \quad \{s_n\}$ is divergent.

Q5. Define the convergence of an infinite series.

Answer :

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if

$$\sum_{n=1}^{\infty} u_n = \sum_{m=1}^{n} u_m = \operatorname*{Lt}_{n\to\infty} S_n = l.$$

Where l is a finite value and S_n is the n^{th} partial sum of the series.

Q6. State the necessary condition for a positive series Σa_n to be convergent.

Answer :

The necessary condition for a positive series Σa_n to be convergent is,

$$\lim_{n\to\infty} a_n = 0$$

i.e, as $n \to \infty$, $a_n \to 0$.

Q7. Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$.

Answer : Dec.-16, Q3

Given series is,

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$$

Let,

$$u_n = \left(1+\frac{1}{n^2}\right)^{n^2}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2} = e$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n = e \neq 0$$

$\therefore$ Σu_n is divergent

Hence, the given series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$ diverges.

Q8. Discuss the convergence of the series $\Sigma \frac{n^2+1}{n^2}$

Answer :

Given that,

$$\sum \frac{n^2+1}{n^2}$$

Let,

$$u_n = \frac{n^2+1}{n^2}$$

Apply limits on both sides,

$$\Rightarrow \operatorname*{Lim}_{n\to\infty} u_n = \operatorname*{Lim}_{n\to\infty} \frac{n^2+1}{n^2}$$

$$\Rightarrow \operatorname*{Lim}_{n\to\infty} u_n = \operatorname*{Lim}_{n\to\infty} \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2} = \operatorname*{Lim}_{n\to\infty}\left(1+\frac{1}{n^2}\right)$$

$$\operatorname*{Lim}_{n\to\infty} u_n = 1$$

$\therefore$ By theorem, if $\operatorname*{Lim}_{n\to\infty} u_n \neq 0$, then $\sum u_n$ is divergent

$\therefore$ $\sum \frac{n^2+1}{n^2}$ is divergent.

Q9. State comparison test.

Answer :

Comparison Test-I

If Σu_n and Σv_n are the series of positive terms such that, $u_n \leq v_n$ ∀ n and series Σv_n is convergent, then other series Σu_n is also convergent.

Comparison Test-II

If Σu_n and Σv_n are the series of positive terms such that, $u_n \geq v_n$ ∀ n and series Σv_n is divergent, then Σu_n is is also divergent.

Comparison Test-III

If Σu_n and Σv_n are the series of positive terms such that $\lim_{n\to\infty}\frac{u_n}{v_n}$ = finite ($\neq 0$), then Σu_n and Σv_n both converge or diverge together.

Q10. Test the convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

Answer :

Given series is,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

Let,

$$u_n = \frac{1}{n^2+1}$$

$$v_n = \frac{1}{n^2}$$

From comparison test,

$$\frac{u_n}{v_n} = \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}}$$

$$\Rightarrow \quad \frac{u_n}{v_n} = \frac{n^2}{n^2+1}$$

$$\Rightarrow \quad \frac{u_n}{v_n} = \frac{n^2}{n^2(1+1/n^2)}$$

$$\Rightarrow \quad \frac{u_n}{v_n} = \frac{1}{1+1/n^2}$$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{1}{1+1/n^2}$$

$$= \frac{1}{1+0}$$

$$= 1 \neq 0$$

But, $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent.

Hence, Σv_n is also convergent by comparison test.

Q11. State P-series test.

Answer :

Let, $\Sigma \frac{1}{n^P}$ be a series, $P \in R$,

Then, $\Sigma \frac{1}{n^P}$

(i) Converges if $p > 1$

(ii) Diverges if $0 < p \leq 1$.

Q12. Find whether the series $\sum \frac{1}{n\sqrt{n^2-1}}$ is convergent or divergent.

Answer :

Given series is,

$$\sum \frac{1}{n\sqrt{n^2-1}} \qquad \text{... (1)}$$

Applying integral test,

$f(x) = \frac{1}{x\sqrt{x^2-1}}$ is a decreasing sequence in [2, ∞]

$$\therefore \quad f(x) = \int_2^t \frac{1}{x\sqrt{x^2-1}} dx$$

$$= \sec^{-1} x \Big|_2^t$$

$$= [\sec^{-1} t - \sec^{-1} 2]$$

$$= \underset{t\to\infty}{Lt} [\sec^{-1} t - \sec^{-1} 2]$$

$$= \sec^{-1} \infty - \sec^{-1} 2$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \to \text{finite}$$

The given series,

$\therefore \quad \Sigma \frac{1}{n\sqrt{n^2-1}}$ is a convergent series.

Q13. State Raabe's test.

Answer :

If Σu_n is a series of positive terms and $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$. Then,

(i) Series is convergent for $k > 1$

(ii) Series is divergent for $k < 1$

(iii) The test fails for $k = 1$.

Q14. State Leibnitz's test.

Answer : June/July-17, Q4

If $\sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ is an alternating series then it is convergent if,

(a) $u_1 \geq u_2 > u_3 \geq ..... \geq u_n \geq u_{n+1}...$

(b) $\lim\limits_{n\to\infty} u_n = 0$.

Q15. Define an alternating series.

Answer :

Alternating Series

A series which contains alternative positive and negative terms is known as alternating series. The expression for an alternating series is given as,

$$u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n-1} u_n + ...$$

(or)

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n$$

Q16. Define the terms (a) absolute convergence and (b) conditional convergence of a series with arbitrary terms.

Dec.-16, Q4

OR

Define the terms (a) absolute convergent series and (b) conditionally convergent series.

Answer : Dec.-17, Q4

Absolute Convergence

Let Σu_n be a series of positive and negative terms. Then Σu_n is said to be absolutely convergent if $|\Sigma u_n|$ is convergent.

Conditionally Convergent

Let Σu_n be a series of positive and negative terms, then Σu_n is said to be conditionally convergent if,

(i) Σu_n is convergent

(ii) $|\Sigma u_n|$ is divergent.

Q17. Show that the series $\sum \frac{\sin n\, x}{n^2}$ converges absolutely.

Answer :

Given series is,

$$\sum \frac{\sin\, n\, x}{n^2}$$

A series $\sum\limits_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if $\sum\limits_{n=1}^{\infty} | a_n |$ converges.

$$\therefore \quad \Sigma \frac{\sin nx}{n^2} \text{ is convergent if } \Sigma \left| \frac{\sin\, n\, x}{n^2} \right| \text{ converges.}$$

$$\text{As } \left| \frac{\sin\, n\, x}{n^2} \right| \leq \frac{1}{n^2} \qquad [\because |\sin n\, x| \leq 1]$$

And as $\Sigma \frac{1}{n^2}$ is a power series with power = 2 which is > 1

$\therefore$ $\Sigma \frac{1}{n^2}$ is convergent, and hence $\Sigma \left| \frac{\sin\, n\, x}{n^2} \right|$ converges

$\therefore$ The given series $\Sigma \frac{\sin\, n\, x}{n^2}$ is absolute convergent.

Q18. Show that the series $\sum \frac{\sin nx}{n^3}$ converges absolutely.

Answer :

Given series is,

$$\sum \frac{\sin nx}{n^3}$$

By definition, a series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum\limits_{n=1}^{\infty} |a_n|$ converges.

$$\sum \frac{\sin nx}{n^3} = \frac{|\sin x|}{1^3} + \frac{|\sin 2x|}{2^3} + \frac{|\sin 3x|}{2^3} + ...$$

$$|u_n| = \frac{|\sin nx|}{n^3}$$

Let,

$$v_n = \frac{1}{n^3}$$

Then,

$$= \frac{|\sin nx|}{n^3} \leq \frac{1}{n^3} \qquad (\because |\sin nx| \leq 1)$$

Also,

$$\Sigma \frac{1}{n^3} \text{ is a power series with power} = 3 > 1.$$

$$\therefore \quad \Sigma \frac{1}{n^3} \text{ is convergent and hence } \Sigma \frac{|\sin nx|}{n^3} \text{ converges.}$$

$$\therefore \quad \text{The given series } \Sigma \frac{|\sin nx|}{n^3} \text{ is absolute convergent.}$$

Q19. Show that series $\sum \frac{\cos nx}{n^2}$ is absolutely convergent.

Answer :

Given series is,

$$\sum \frac{\cos nx}{n^2}$$

A series $\sum\limits_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if $\sum\limits_{n=1}^{\infty} | a_n |$ converges.

$$\therefore \quad \sum \frac{\cos nx}{n^2} \text{ is convergent if } \sum \left| \frac{\cos nx}{n^2} \right| \text{ converges.}$$

$$\text{As } \left| \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2}$$

$\Sigma \frac{1}{n^2}$ is a power series with power ‘2’, which is > 1.

$\therefore$ $\Sigma \frac{1}{n^2}$ is convergent, and hence $\sum \left| \frac{\cos nx}{n^2} \right|$ converges.

$\therefore$ The given series $\sum \frac{\cos nx}{n^2}$ is absolutely convergent.

Q20. Discuss the convergence of the series $\sum \frac{1}{n^2}$.

Answer :

Given that,

$$\sum \frac{1}{n^2}$$

Here,

$$\sum \frac{1}{n^P} = \sum \frac{1}{n^2}$$

$P = 2 > 1$ [$\because$ $P > 1$ the series is convergent by P-test]

$\therefore$ $\sum \frac{1}{n^P}$ is convergent by P-test

Hence, $\sum \frac{1}{n^2}$ is convergent by P-test.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

1.1 SEQUENCES, SERIES, GENERAL PROPERTIES OF SERIES, SERIES OF POSITIVE TERMS

Q21. Define the following terms,

(a) Sequence

(b) Limit

(c) Convergent sequence

(d) Divergent sequence

(e) Bounded sequence

(f) Monotonic sequence.

Answer :

(a) Sequence

For answer refer Unit-3, Q1.

(b) Limit

For answer refer Unit-3, Q2.

(c) Convergent Sequence

For answer refer Unit-3, Q3.

(d) Divergent Sequence

For answer refer Unit-3, Q4.

(e) Bounded Sequence

A sequence of the form $u_n < k$ (where k is any number) is known as bounded sequence.

(f) Monotonic Sequence

A sequence which increases or decreases with respect to $u_{n+1} \geq u_n$ (or) $u_{n+1} \leq u_n$ is termed as monotonic sequence.

Examples

(i) 1, 4, 7, 10, ...

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

A sequence which satisfies both monotonic and bounded sequence is known as convergent sequence.

Q22. Define and explain about oscillatory sequence.

Answer :

Oscillatory Sequence

Let a_n be a sequence. If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, then a_n is called as an “oscillatory sequence”.

There are two types of oscillatory sequences. They are,

(i) Oscillate finitely and

(ii) Oscillate infinitely.

(i) Oscillate Finitely

If a bounded sequence does not converge, then it is said to oscillate finitely.

For example,

Consider the sequence $\{(-1)^n\}$.

Let,

$$a_n = (-1)^n$$

It is a bounded sequence,

$$\Rightarrow \quad \underset{n \to \infty}{Lt}\ a_{2n} = \underset{n \to \infty}{Lt}\ (-1)^{2n} = 1$$

and

$$\Rightarrow \quad \underset{n \to \infty}{Lt}\ a_{2n+1} = \underset{n \to \infty}{Lt}\ (-1)^{2n+1} = -1$$

Where,

$$n = 1, 2, 3, ... \infty.$$

$\therefore$ For the given sequence $\underset{n \to \infty}{Lt}\ \{a_n\}$ does not exist.

$\therefore$ The given sequence does not converge. Hence, the given sequence oscillates finitely.

(ii) Oscillate Infinitely

If an unbounded sequence does not diverge, then it is said to oscillate infinitely.

For example,

Consider, the sequence $\{(-1)^n.n\}$.

Let,

$$a_n = (-1)^n.n$$

It is an unbounded sequence,

$$\Rightarrow \quad \underset{n \to \infty}{Lt}\ a_{2n} = \underset{n \to \infty}{Lt}\ (-1)^{2n}.2n$$

$$= \underset{n \to \infty}{Lt}\ 2n = +\infty$$

$$\underset{n \to \infty}{Lt}\ a_{2n} = +\infty$$

and

$$\underset{n \to \infty}{Lt}\ a_{2n+1} = \underset{n \to \infty}{Lt}\ (-1)^{2n+1}.2n+1$$

$$= \underset{n \to \infty}{Lt}\ -(2n+1)$$

$$= -\infty$$

$$\therefore \quad \underset{n \to \infty}{Lt}\ a_{2n+1} = -\infty$$

$\therefore$ The given sequence does not diverge.

Hence, the given sequence oscillates infinitely.

Q23. Write about the following,

(i) General properties of series

(ii) Series of positive terms.

Answer :

(i) General Properties of Series

Property 1

The addition or removal of finite number of terms from series, does not have any affect on the nature of the series (i.e., convergence or divergence of an infinite series).

Property 2

For a series containing positive and negative terms and all the positive terms are convergent, then the series remains convergent. Moreover, the presence of negative terms does not affect the nature of series.

Property 3

The multiplication of finite number to the terms of infinite series does not affect the nature of the series.

(ii) Series of Positive Terms

An infinite series which contains all the positive terms after few particular terms is called series of positive terms.

Example: – 1 – 2 – 3 + 4 + 5 + 6 + 7 + 8 + ...

Condition for Convergence

The necessary condition for convergence of a positive term series Σu_n is given as,

$$\Sigma \underset{n \to \infty}{Lim}\, u_n = 0$$

Q24. Define the convergence of an infinite series. Show that the nth term of a convergent series tends to zero. Is the converse true?

Answer :

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if $\sum_{n=1}^{\infty} u_n = \sum_{m=1}^{\infty} u_m = \underset{n \to \infty}{\text{Lt}}\ S_n = l$, Where l is a finite value and unique and S_n is the n^{th} partial sum of the series.

If $\underset{n \to \infty}{\text{Lt}}\ S_n$ does not exist, then the series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

If the series Σu_n is convergent, then $\underset{n \to \infty}{\text{Lt}}\ u_n = 0$. It is only a necessary condition and not sufficient so, the converse is not true.

If $\sum_{n=1}^{\infty} u_n$ is a sequence and if S_n and S_{n-1} are its nth and $(n-1)^{\text{th}}$ partial sums.

Then, $S_n - S_{n-1} = u_n$

$$\underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}}\ (S_n - S_{n-1}) = 1 - 1 = 0$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\ u_n = 0$$

$\therefore$ Converse is not true.

1.2 COMPARISON TESTS, TESTS OF CONVERGENCE D'ALEMBERT'S RATIO TEST

Q25. State and prove comparison test.

Answer :

Comparison Test I

If Σu_n and Σv_n are the series of positive terms such that, $u_n \leq v_n$ ∀ n and series Σv_n is convergent, then other series Σu_n is also convergent.

Proof

Consider two series of positive terms,

$$u_n = u_1 + u_2 + u_3 + ... + u_n \text{ and}$$

$$v_n = v_1 + v_2 + v_3 + ... + v_n$$

For $u_n \leq v_n$

$$\lim_{n\to\infty} u_n < \lim_{n\to\infty} v_n$$

Since, Σv_n is convergent, $\lim_{n\to\infty} v_n$ is finite

Therefore, $\lim_{n\to\infty} u_n$ also has a finite value and it is also a convergent series.

Comparison Test II

If Σu_n and Σv_n are the series of positive terms such that, $u_n \geq v_n$ ∀ n and series Σv_n is divergent, then Σu_n also divergent.

Proof

Consider two series of positive terms,

$$u_n = u_1 + u_2 + u_3 + ... + u_n \text{ and}$$

$$v_n = v_1 + v_2 + v_3 + ... + v_n$$

For $u_n \geq v_n$

$$\lim_{n\to\infty} u_n \geq \lim_{n\to\infty} v_n$$

Since, Σv_n is divergent, $\lim_{n\to\infty} v_n = \infty$

Therefore, $\lim_{n\to\infty} u_n = \infty$ and Σu_n also divergent

Comparison Test III

If Σu_n and Σv_n are the series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite ($\neq 0$), then Σu_n and Σv_n both converge or diverge together.

Since,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l$$

By the definition of limit,

$$\left|\frac{u_n}{V_n} - l\right| < \varepsilon \text{ for } n \geq m$$

Where,

ε – Positive number

(or)

$$-\varepsilon < \frac{u_n}{v_n} - l < \varepsilon \quad ; \quad \text{for } n \geq m$$

(or)

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon\,; \quad \text{for } n \geq m$$

By eliminating the first '*m*' terms of both the series,

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \ \text{ for all } n \qquad \text{... (1)}$$

Thus, there exist two cases.

Case(i): When Σv_n is Convergent

If Σv_n is convergent, then,

$$\lim_{n\to\infty} (v_1 + v_2 + v_3 + ...+ v_n) = k \qquad \text{... (2)}$$

From equation (1),

$$\frac{u_n}{v_n} < l + \varepsilon$$

$$\Rightarrow \quad u_n < (1 + \varepsilon)\, v_n \quad \text{For all } n$$

$$\therefore \quad \lim_{n\to\infty} (u_1 + u_2 + ...+ u_n) < (l + \varepsilon) \lim_{n\to\infty} (v_1 + v_2 + ...+ v_n) = (l + \varepsilon)k \quad [\because \text{ From equation (2)}]$$

$\therefore$ Σu_n is also convergent

Case(ii): When Σv_n is Divergent

If Σv_n is divergent then,

$$\lim_{n\to\infty} (v_1 + v_2 + v_3 + ...+ v_n) = \infty \qquad \text{... (3)}$$

From equation (1),

$$l - \varepsilon < \frac{u_n}{V_n}$$

$$\Rightarrow \quad u_n > (1 - \varepsilon) v_n \quad \text{For all } n$$

$$\therefore \quad \lim_{n\to\infty} (u_1 + u_2 + ...+ u_n) > (l - \varepsilon) \lim_{n\to\infty} (v_1 + v_2 + ...+ v_n) \to \infty \quad [\because \text{ From equation (3)}]$$

Therefore, Σu_n is also divergent.

Q26. Test the series $\sum\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$ for convergence.

Answer :

Given series is,

$$u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$$

$$u_n = \left[\sqrt{n^4+1} - \sqrt{n^4-1}\right] \times \frac{\left[\sqrt{n^4+1} + \sqrt{n^4-1}\right]}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$u_n = \frac{\left(\sqrt{n^4+1}\right)^2 - \left(\sqrt{n^4-1}\right)^2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$u_n = \frac{n^4+1-n^4+1}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$v_n = \frac{\text{Highest power of } 'n' \text{ in numerator}}{\text{Highest power of } 'n' \text{ in denominator}}$$

$$\Rightarrow \quad v_n = \frac{n^0}{\sqrt{n^4}}$$

$$\Rightarrow \quad v_n = \frac{1}{n^2} \Rightarrow P = 2 > 1$$

$\therefore \quad \sum v_n$ is convergent by *P*-Test

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\dfrac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}}{\dfrac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2\left[\sqrt{1+\dfrac{1}{n^4}}+\sqrt{1-\dfrac{1}{n^4}}\right]}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{1+1} = \frac{2}{2} = 1 \neq 0$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} \neq 0$$

$\Rightarrow \quad \sum u_n, \sum v_n$ are convergent by limit comparison test.

$\therefore \quad \sum u_n = \sum\left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]$ is convergent.

Q27. Test the convergence of the series $\sum_{n=1}^{\infty} \dfrac{1}{x^n + x^{-n}}$

Answer :

Given series is,

$$\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$$

Case (i) : When x > 1

Comparing given series Σu_n with $\Sigma v_n = \Sigma x^{-n}$,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{x^n + x^{-n}} . x^n$$

$$= \lim_{n\to\infty} \frac{1}{(x^n + x^{-n})\, x^{-n}}$$

$$= \lim_{n\to\infty} \frac{1}{\dfrac{x^n}{x^n} + x^{-2n}}$$

$$= \lim_{n\to\infty} \frac{1}{1 + x^{-2n}}$$

$$= \frac{1}{1+0} = 1 \qquad [\because x^{-2n} = 0 \text{ as } n \to \infty]$$

Σv_n is convergent.

$\therefore \quad \Sigma u_n$ is also convergent.

Case (ii): When x < 1

Comparing the given series Σu_n with $\Sigma v_n = \Sigma x^n$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{x^n + x^{-n}} . \frac{1}{x^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{x^{2n} + \dfrac{x^n}{x^n}}\right) = \lim_{n\to\infty} \left(\frac{1}{x^{2n}+1}\right)$$

$$= \frac{1}{0+1} = 1 \qquad [\because x^{2n} = 0 \text{ as } n \to \infty]$$

Σv_n is convergent.

$\therefore \quad \Sigma u_n$ is also convergent.

Case (iii): When x = 1

$$\Sigma u_n = \frac{1}{2} + \frac{1}{2} + \ldots\ldots + \infty$$

Which is divergent.

$\therefore \quad \Sigma u_n$ is divergent when $x = 1$ and convergent for $x < 1$ and $x > 1$.

Q28. Test for convergence of the series

$$\mathbf{\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots\ldots.}$$

Answer :

Given series is,

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots..$$

Let, $\quad u_n = \dfrac{n+1}{n^p}$ and $v_n = \dfrac{1}{n^{p-1}}$

Then,

$$\frac{u_n}{v_n} = \frac{n+1}{n^p}.n^{p-1}$$

$$= \frac{n\left(1+\frac{1}{n}\right)}{n^p}.n^{p-1}$$

$$= \frac{1+\frac{1}{n}}{n^{p-1}}.n^{p-1}$$

$$\therefore \quad \frac{u_n}{v_n} = 1 + \frac{1}{n}$$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)$$

$$= \left(1+\frac{1}{\infty}\right) = (1 + 0) = 1 \neq 0$$

$\therefore$ By comparison test Σu_n and Σv_n are convergent.

Q29. Test for convergence of $\sum \frac{(n+1)\,(n+2)}{n^3\sqrt{n}}$.

Answer :

Given series is,

$$u_n = \frac{(n+1)(n+2)}{n^3\sqrt{n}}$$

$$\Rightarrow \quad u_n = \frac{n\left(1+\frac{1}{n}\right)n\left(1+\frac{2}{n}\right)}{n^3\sqrt{n}}$$

$$= \frac{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^3\sqrt{n}}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n\sqrt{n}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^{3/2}}$$

Let, $v_n = \frac{1}{n^{3/2}} = \frac{1}{n^P}$ where $P = \frac{3}{2} > 1$

$\therefore \quad v_n$ is convergent by P-test.

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^{3/2}}}{\frac{1}{n^{3/2}}}$$

$$= \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \text{ as } n \to \infty,\ \frac{1}{n} \to 0$$

$$= (1 + 0)\,(1 + 0) = 1 \text{ (finite)}$$

$\therefore$ By comparison test, u_n and v_n convergent or divergent together.

$\therefore \quad v_n$ is convergent, $\frac{u_n}{v_n}$ is also convergent.

$\therefore$ The given series $\sum \frac{(n+1)(n+2)}{n^3\sqrt{n}}$ is also convergent.

Q30. Discuss the convergence of the series $\sum\left[\frac{\sqrt{n+1}-\sqrt{n}}{n^2}\right]$.

Answer : Dec.-16, Q17(a)

Given series is,

$$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^2}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{n^2} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{\left(\sqrt{n+1}\right)^2-\left(\sqrt{n}\right)^2}{n^2\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \frac{n+1-n}{n^2\left(\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)\right)}$$

$$= \frac{1}{n^2.n^{\frac{5}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{n^{\frac{5}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$$v_n = \frac{\text{Highest Power of 'n' in numerator}}{\text{Highest Power of 'n' in denominator}}$$

Highest power of n in denominator

$$= \frac{n^0}{n^{\frac{5}{2}}} = \frac{1}{n^{\frac{5}{2}}}$$

$$\therefore \quad v_n = \frac{1}{n^{\frac{5}{2}}}$$

By p-series, $p = \frac{5}{2} > 1$; Σv_n is convergent.

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{lt} \frac{\frac{1}{n^{\frac{5}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)}}{{\frac{1}{n}}^{\frac{5}{2}}}$$

$$= \underset{n\to\infty}{lt} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{1}{\sqrt{1+0}+1}$$

$$= \frac{1}{1+1} = \frac{1}{2} \neq 0$$

$\therefore \quad \Sigma u_n$; Σv_n are convergent.

$\therefore \quad \frac{\sqrt{n+1}-\sqrt{n}}{n^2}$ is convergent.

Q31. Test for convergence of the series,

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+.....\infty.$$

Answer :

Given series is,

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+...\infty$$

i.e., $\Sigma u_n = \sum \frac{(2n+1)}{n(n+1)(n+2)}$

$$\Rightarrow \quad u_n = \frac{2n+1}{n(n+1)(n+2)} \qquad ... (1)$$

Numerator is a linear factor and denominator is a polynomial of degree 3.

$$\therefore \ \Sigma v_n = \sum \frac{n}{n^3} = \sum \frac{1}{n^2}$$

$$\Rightarrow \quad v_n = \frac{1}{n^2} \qquad ... (2)$$

Dividing equation (1) by equation (2),

$$\Rightarrow \quad \frac{u_n}{v_n} = \frac{(2n+1)}{n(n+1)(n+2)} \times \frac{n^2}{1}$$

$$= \frac{n^2.n\left(2+\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\frac{u_n}{v_n} = \frac{2+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{2+0}{(1+0)(1+0)}$$

$$= \frac{2}{1} = 2 \neq 0$$

Therefore, Σu_n and Σv_n converge or diverge together by comparison test.

$\Sigma v_n = \Sigma \frac{1}{n^2}$ is a P-series with $P = 2 > 1$ $\left(\because \Sigma v_n = \Sigma \frac{1}{n^p} \text{ is a } P\text{-series}\right)$

A P-series with $P > 1$ converges

$\therefore$ Σv_n is convergent

Similarly, Σu_n is also convergent.

Q32. State and prove D-Alembert's ratio test.

Answer :

If Σu_n is a series of positive terms such that,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$$

(i) If $\lambda < 1$, Σu_n is convergent

(ii) If $\lambda > 1$, Σu_n is divergent

(iii) If $\lambda = 1$, test fails.

Case (i) : When $\mathbf{\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda < 1}$

From the definition of a limit, a positive r(< 1) can be determined such that,

$\frac{u_{n+1}}{u_n} < r$; for all $n > m$

By eliminating the first 'm' terms, the series is given as,

$u_1 + u_2 + u_3 + ...$

Where the common ratio is given as,

$$\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r .....$$

Then, $u_1 + u_2 + u_3 + u_4 + ... \infty$

$$= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \times \frac{u_2}{u_1} + \frac{u_4}{u_3} \times \frac{u_3}{u_2} \times \frac{u_2}{u_1} + ...\infty\right) < (1 + r + r^2 + r^3 + ... \infty)$$

$$= \frac{u_1}{1-r} \qquad \left(\because \text{Sum of Geometric series, } S_n = \frac{a(1-r)^n}{1-r}\right)$$

$\therefore$ Σu_n is convergent.

Case (ii): Where $\mathbf{\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda > 1}$

From the definition of limit, 'm' value can be determined, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $\geq m$

By eliminating the first 'm' terms the series is given as,

$u_1 + u_2 + u_3 + ...$

Where,

$$\frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1 .....$$

$$\therefore \quad u_1 + u_2 + u_3 + u_4 + ... u_n = u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} . \frac{u_2}{u_1} + ...\right) \geq u_1 (1 + 1 + 1 + ...n)$$

$$= nu_1$$

$$\therefore \lim_{n\to\infty} (u_1 + u_2 + ... + u_n) \geq \lim_{n\to\infty} (nu_1)$$

As $\lim_{n\to\infty} (nu_1) \to \infty$, Σu_n is said to be a divergent series

Case (iii) : When $\lambda = 1$

Consider a series $\Sigma u_n = \Sigma \frac{1}{n^p}$

$$\lambda = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[\frac{1}{(n+1)^p} n^p\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^p\left(1+\frac{1}{n}\right)^p} n^p\right] = \frac{1}{\left(1+\frac{1}{\infty}\right)^p}$$

$$= \frac{1}{(1+0)^p} = \frac{1}{1} = 1$$

Therefore, the series $\sum \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p < 1$.

For $p = 1$, it is not possible to find the nature of the series thus test fails at $\lambda = 1$.

Q33. Examine the convergence of the series $\frac{1}{1^p} + \frac{x}{3^p} + \frac{x^2}{5^p} + \ldots + \frac{x^{n-1}}{(2n-1)^p} + \ldots$

Answer :

Given series is,

$$\frac{1}{1^P} + \frac{x}{3^P} + \frac{x^2}{5^P} + \ldots . + \frac{x^{n-1}}{(2n-1)^P}$$

Consider,

$$\Sigma u_n = \Sigma \frac{x^{n-1}}{(2n-1)^p}$$

$$\Rightarrow \quad u_n = \frac{x^{n-1}}{(2n-1)^p} = \frac{x^{n-1}}{n^p\left(2-\frac{1}{n}\right)^p}$$

Then, $u_{n+1} = \frac{x^{n+1-1}}{(2(n+1)-1)^p} = \frac{x^n}{(2n+1)^p} = \frac{x^n}{n^p\left(2+\frac{1}{n}\right)^p}$

By D-Alembertz ratio test,

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{\frac{x^n}{n^p(2+1/n)^p}}{\frac{x^{n-1}}{n^p(2-1/n)^p}} = \frac{x.\left(2-\frac{1}{n}\right)^p}{\left(2+\frac{1}{n}\right)^p}$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{x\left(2-\frac{1}{n}\right)^p}{\left(2+\frac{1}{n}\right)^p} = \frac{x(2-0)^p}{(2+0)^p} = \frac{x(2^p)}{2^p} = x$$

$\therefore$ Σu_n converges when $x < 1$, diverges when $x > 1$ and fails when $x = 1$.

$\therefore$ By limit comparison test for $n = 1$.

$$v_n = \frac{1}{n^p}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \frac{\dfrac{1}{n^p(2-1/n)^p}}{\dfrac{1}{n^p}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{(2-1/n)^p}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{(2-1/n)^p} = \frac{1}{2} \text{ (finite)}$$

Σu_n converges when $x \leq 1$ and diverges when $x > 1$.

Q34. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

Answer :

Given that,

$$\sum u_n = \sum_{n=1}^{\infty} \frac{n^2}{3^n}$$

$$\Rightarrow \quad u_n = \frac{n^2}{3^n}$$

$$\Rightarrow \quad u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$$

By applying D-Alembert's ratio test,

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{(n+1)^2}{3^{n+1}}}{\dfrac{n^2}{3^n}}$$

$$= \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2}$$

$$= \frac{n^2\left(1+\frac{1}{n}\right)^2}{3^n.3} \cdot \frac{3^n}{n^2} = \frac{n^2\left(1+\frac{1}{n}\right)^2}{3n^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{\left(1+\frac{1}{n}\right)^2}{3}$$

Applying limit on both sides,

$$\underset{n\to\infty}{\text{Lim}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lim}} \frac{\left(1+\frac{1}{n}\right)^2}{3}$$

$$= \frac{(1+0)^2}{3} = \frac{1^2}{3} \quad \left(\because n \to \infty, \frac{1}{n} \to 0\right)$$

$$= \frac{1}{3} < 1$$

$$\therefore \quad \underset{n\to\infty}{\text{Lim}} \frac{u_{n+1}}{u_n} < 1$$

$\therefore$ By D'Alembert's ratio test, the given series $\sum u_n$ is convergent.

Q35. Test for convergence of $\sum \frac{x^n}{n(n-1)(n-2)}$.

Answer :

Given that,

$$\sum \frac{x^n}{n(n-1)\,(n-2)}$$

Let,

$$u_n = \frac{x^n}{n(n-1)\,(n-2)}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)(n)(n-1)}$$

Applying D-Alembert's ratio-test

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{x^{n+1}}{n(n-1)(n+1)}}{\dfrac{x^n}{n(n-1)(n-2)}}$$

$$= \frac{\dfrac{x}{n+1}}{\dfrac{1}{n-2}} = \frac{x\,(n-2)}{(n+1)}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{xn\left(1-\frac{2}{n}\right)}{n\left(1+\frac{1}{n}\right)} = x$$

$\therefore$ By D'Alembert's ratio test,

$\Sigma\, u_n$ converges when $x < 1$, diverges when $x > 1$ and fails when $x = 1$

When $x = 1$, by comparison test,

i.e., $$u_n = \frac{1}{n(n-1)\,(n-2)}$$

$$v_n = \frac{1}{n^3} = \frac{1}{n^p}\,; P = 3 > 1$$

$\therefore$ v_n is convergent by P-test.

$$\frac{u_n}{v_n} = \frac{\dfrac{1}{n(n-1)(n-2)}}{\dfrac{1}{n^3}} = \frac{\dfrac{1}{n^3}\left(1-\dfrac{1}{n}\right)\left(1-\dfrac{2}{n}\right)}{\dfrac{1}{n^3}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \frac{(1)(1)}{1} = 1 \text{ (finite)}$$

$\therefore$ v_n converges.

$\therefore$ $\Sigma u_n = \sum \frac{x^n}{n(n-1)\,(n-2)}$ converges when $x \leq 1$ and diverges when $x > 1$.

Q36. Discuss the convergence of the exponential series $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

Answer :

Given series $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots .$

Let the given series be,

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$u_n = \frac{x^{n-1}}{(n-1)!}\,;\ u_{n+1} = \frac{x^n}{n!}$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{x^{n-1}}{(n-1)!}\ \frac{n!}{x^n}$$

$$= \frac{n}{|x|} \text{ for } x \neq 0$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n}{|x|} = \infty > 1\ \forall\ x \neq 0$$

$\therefore$ By D-Alembert's ratio test, the series $\sum_{n=1}^{\infty} |u_n|$ is convergent for $x \neq 0$.

When $x = 0$ the given series is,

$1 + 0 + 0 + \ldots .$ which is convergent.

Hence, $\sum_{n=1}^{\infty} |u_n|$ is convergent $\forall\ x$.

$\therefore$ The given series is absolutely convergent for all x.

Q37. Discuss the convergence of the series $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}}{\sqrt{n^2+1}} x^n,\ x > 0.$

Answer : June/July-17, Q17(a)

Given series is,

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}}\ x^n$$

Let, $u_n = \dfrac{\sqrt{n}}{\sqrt{n^2+1}} x^n = \sqrt{\dfrac{n}{n^2+1}}\,.x^n$

$$\Rightarrow \quad u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} x^{n+1}$$

$$= \sqrt{\frac{n+1}{(n+1)^2+1}}\, x^{n+1}$$

By applying D-Alembert's ratio test

$$\frac{u_n}{u_{n+1}} = \frac{\sqrt{\dfrac{n}{n^2+1}}\, x^n}{\sqrt{\dfrac{n+1}{(n+1)^2+1}}\, x^{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1}\ \frac{n^2+2n+2}{n^2+1}}\,.\frac{1}{x}$$

$$\Rightarrow \underset{n\to\infty}{Lim} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lim} \sqrt{\frac{1}{1+\dfrac{1}{n}}\ \frac{1+\dfrac{2}{n}+\dfrac{2}{n^2}}{1+\dfrac{1}{n^2}}}\,.\frac{1}{x} = \frac{1}{x}(1) = \frac{1}{x}$$

$\therefore$ By D-Alembert's ratio test,

Σu_n converges if $\dfrac{1}{x} > 1$ i.e., $x < 1$

Σu_n diverges if $\dfrac{1}{x} < 1$ i.e., $x > 1$ and the ratio test fails when $x = 1$

When $x = 1$,

$$u_n = \sqrt{\frac{n}{n^2+1}}\,(1)^n$$

$$= \sqrt{\frac{n}{n^2+1}}$$

$$= \sqrt{\frac{n}{\left(1+\dfrac{1}{n^2}\right)n^2}}$$

$$= \sqrt{\frac{1}{n}\left(\frac{1}{1+n^2}\right)}$$

$$= \frac{1}{\sqrt{n}}\sqrt{\frac{1}{1+n^2}}$$

Let, $v_n = \dfrac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{Lim} \frac{u_n}{v_n} = \underset{n\to\infty}{Lim} \frac{\dfrac{1}{\sqrt{n}}\sqrt{\dfrac{1}{1+\dfrac{1}{n^2}}}}{\dfrac{1}{\sqrt{n}}} = \underset{n\to\infty}{Lim} \sqrt{\frac{1}{1+\dfrac{1}{n^2}}} = 1$$

$\therefore \quad \underset{n\to\infty}{Lim} \dfrac{1}{1+\dfrac{1}{n^2}} \neq 0$ i.e., it is a finite value.

According to comparison test, Σu_n and Σv_n converge or diverge together.

As $\Sigma v_n = \Sigma \dfrac{1}{\sqrt{n}}$ which is of the form $\Sigma \dfrac{1}{n^p}$ with $p = \dfrac{1}{2} < 1$

$\therefore \quad \Sigma v_n$ diverges

$\Rightarrow \quad \Sigma u_n$ diverges

Therefore, the series converges when $x < 1$ and diverges when $x \geq 1$.

Q38. Test for convergence of the series,

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+......$$

Answer :

Given series is,

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+......$$

i.e.,
$$u_n=\frac{x^{2n-2}}{\sqrt{n}(n+1)} \qquad ...\ (1)$$

Then,
$$u_{n+1}=\frac{x^{2(n+1)-2}}{((n+1)+1)\sqrt{n+1}}$$

$$u_{n+1}=\frac{x^{2n+2-2}}{(n+2)\sqrt{n+1}}$$

$$u_{n+1}=\frac{x^{2n}}{(n+2)\sqrt{n+1}} \qquad ...\ (2)$$

Dividing equation (2) by equation (1),

$$\frac{u_{n+1}}{u_n}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}\cdot\frac{\sqrt{n}(n+1)}{x^{2n-2}}$$

$$=\frac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}(x^{2n+2-2n})$$

$$=\left(\frac{n+1}{n+2}\left(\frac{n}{n+1}\right)^{1/2}\right)x^2$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}.\frac{n^{1/2}}{n^{1/2}\left(1+\frac{1}{n^{1/2}}\right)}\right]x^2=x^2$$

Hence, Σu_n converges when $x^2<1$ and

Σu_n diverges when $x^2>1$

When,

$$x^2=1$$

$$u_n=\frac{1}{(n+1)\sqrt{n}}$$

$$=\frac{1}{(n+1)n^{1/2}}$$

$$=\frac{1}{n^{3/2}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)}$$

and,

$$v_n=\frac{1}{n^{3/2}}$$

$$\therefore\ \underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\frac{1}{n^{3/2}}\cdot\frac{n^{3/2}}{\left(1+\frac{1}{n}\right)}=1$$

$\therefore$ Σv_n is convergent series, and

Σu_n is also convergent.

$\therefore$ The given series converges when and $x^2>1$ it diverges, when $x^2>1$.

1.3 CAUCHY'S n^th ROOT TEST, RAABE'S TEST, LOGARITHMIC TEST

Q39. Write about Cauchy's root test.

Answer :

If Σu_n is a positive series such that $\lim\limits_{n\to\infty} u_n^{1/n}=\lambda$ then

(i) Σu_n is convergent if $\lambda<1$

(ii) Σu_n is divergent if $\lambda>1$

(iii) Test fails if $\lambda=1$

Case (i): $\lambda<1$

If $\lim\limits_{n\to\infty} u_n^{1/n}=\lambda<1$

From the definition of limit, the value of a positive number r ($\lambda<r<1$) can be determined, such that,

$(u_n)^{1/n}<r$ for all $n>m$

$\Rightarrow$ $u_n<r^n$ for all $n>m$

$\because$ $r<1$ then Σr^n is convergent.

Hence, by comparison test, Σu_n is also convergent.

Case (ii): $\lambda>1$

If $\lim\limits_{n\to\infty}(u_n)^{1/n}=\lambda>1$

From the definition of limit a positive number 'm' can be determined, such that,

$(u_n)^{1/n}>1$ for all $n>m$

(or)

$u_n>1$ for all $n>m$

By eliminating the first m terms, the series is given as,

$u_1+u_2+u_3+....$ where $u_1>1$, $u_2>1$, $u_3>1$

$\therefore$ $u_1+u_2+u_3+......+u_n>n$

And $\lim\limits_{n\to\infty}(u_1+u_2+......+u_n)\to\infty$

Thus, Σu_n is divergent.

Case (iii): $\lambda = 1$

When $\lambda = 1$ Cauchy's root test fails and other tests are to be applied.

Q40. Discuss the convergence of the series, $\sum \frac{[(n+1)x]^n}{n^{n+1}}$.

Answer :

Given series is,

$$\sum \frac{[(n+1)x]^n}{n^{n+1}}$$

i.e., $$u_n = \frac{((n+1)x)^n}{n^{n+1}}$$

$$\underset{n\to\infty}{Lim}\,(u_n)^{1/n} = \underset{n\to\infty}{Lim}\left[\frac{[(n+1)x]^n}{n^n.n^1}\right]^{1/n}$$

$$= \underset{n\to\infty}{Lim}\frac{(n+1)x}{n^{1+\frac{1}{n}}}$$

$$= \underset{n\to\infty}{Lim}\frac{n\left(1+\frac{1}{n}\right)x}{n^{1+\frac{1}{n}}}$$

$$= \underset{n\to\infty}{Lim}\frac{n\left(1+\frac{1}{n}\right)x}{n^{\frac{1}{n}}}$$

$$= \frac{(1+0)x}{n^0} = x$$

$$\underset{n\to\infty}{Lim}\,(u_n)^{1/n} = x$$

By Cauchy's root test, Σu_n

(i) Converges, if $x < 1$

(ii) Diverges, if $x > 1$

(iii) At $x = 1$, Root test fails then

By applying limit comparison test,

$$u_n = \left(\frac{((n+1)x)^n}{n^{n+1}}\right)$$

$$u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n.n^1}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n}{n} \quad \text{... (1)}$$

$$v_n = \frac{1}{n} \quad \text{... (2)}$$

Dividing equation (1) by equation (2) and applying limit.

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\frac{\frac{\left(1+\frac{1}{n}\right)^n}{n}}{(1/n)}$$

$$= \underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{1}{n}\right)^n}{n}.n = e \neq 0$$

If $\underset{n\to\infty}{\text{Lt}}\, u_n \neq 0$ then Σu_n is divergent.

$\therefore$ Σu_n is divergent for $x = 1$ ... (2)

Combining equations (1) and (2),

Hence, Σu_n

(i) Converges if $x < 1$

(ii) Diverges if $x \geq 1$.

Q41. Test the series $\sum\left(1+\frac{1}{n}\right)^n$ for convergence.

Answer :

Let, $$u_n = \left(1+\frac{1}{n}\right)^n \quad \text{... (1)}$$

Taking n^{th} root on both sides of equation (1),

$$\sqrt[n]{u_n} = \sqrt[n]{\left(1+\frac{1}{n}\right)^n}$$

$$\Rightarrow (u_n)^{1/n} = \left[\left(1+\frac{1}{n}\right)^n\right]^{1/n}$$

$$\Rightarrow (u_n)^{1/n} = 1+\frac{1}{n} \quad \text{... (2)}$$

Applying limits on both sides of equation (2),

$$\underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)$$

$$= 1+\frac{1}{\infty}$$

$$= 1+0 \qquad \left[\because \frac{1}{\infty} = 0\right]$$

$$\underset{n\to\infty}{\text{Lt}}\,(u_n)^{1/n} = 1$$

Therefore, according to Cauchy's root test,

$$\left.\begin{array}{l}\Sigma u_n \text{ is convergent if } x < 1 \text{ and} \\ \Sigma u_n \text{ is divergent if } x > 1\end{array}\right\} \quad \text{... (3)}$$

Here, $x = 1$, Cauchy's root test fails.

Now, applying some other test such that,

$$u_n = \left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \quad u_n = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{-n}}\right] \qquad \ldots (4)$$

Applying n^{th} root test on equation (4),

$$\Rightarrow \quad (u_n)^{1/n} = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{-n}}\right]^{1/n}$$

$$\Rightarrow \quad (u_n)^{1/n} = \left[\frac{1}{\left(1+\frac{1}{n}\right)^{-n\left(\frac{1}{n}\right)}}\right]$$

$$\Rightarrow \quad (u_n)^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^{-1}} \qquad \ldots (5)$$

Taking limits on both sides of equation (5),

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{\left(1+\frac{1}{n}\right)^{-1}}$$

$$= \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)$$

$$= 1+\frac{1}{\infty}$$

$$= 1 + 0$$

$$= 1$$

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} \neq 0$$

If $\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} \neq 0$ then Σu_n is divergent.

Therefore, the given series is divergent.

Since, $\Sigma u_n = 1 \qquad \ldots (6)$

Thus, from equations (3) and (6),

Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

Q42. Test for convergence of the series $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

Answer :

$$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$$(u_n)^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{\frac{-n^{3/2}}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}}$$

$$(u_n)^{1/n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\underset{n\to\infty}{Lt}\ (u_n)^{1/n} = \underset{n\to\infty}{Lt}\ \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{(1+0)^{\sqrt{n}}} = \frac{1}{e} < 1.$$

$\therefore$ By Cauchy's n^{th} root test, Σu_n converges.

Q43. Test the convergence of the series $\left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \ldots$

Answer :

Given series is,

$$\left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \ldots$$

$$= \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)x^n$$

$$\text{Let, } u_n = \left(\frac{n+1}{n+2}\right)x^n$$

$$(u_n)^{1/n} = \left(\frac{n+1}{n+2}\right)^{1/n}.x$$

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ \frac{n^{1/n}\left(1+\frac{1}{n}\right)^{1/n}}{n^{1/n}\left(1+\frac{2}{n}\right)^{1/n}}x$$

$$= \underset{n\to\infty}{\text{Lt}}\ \frac{\left(1+\frac{1}{n}\right)^{1/n}}{\left(1+\frac{2}{n}\right)^{1/n}}x = x$$

∴ By Cauchy's n^{th} root test, the series converges if $x < 1$, diverges if $x > 1$ ... (1)

For $x = 1$,

$$u_n = \left(\frac{n+1}{n+2}\right)$$

$$\operatorname*{Lt}_{n\to\infty}(u_n) = \operatorname*{Lt}_{n\to\infty}\left[\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right] = 1 \neq 0 \qquad \text{... (2)}$$

∴ From equations (1) and (2), the given series converges if $x < 1$ and diverge if $x \geq 1$.

Q44. Discuss the convergence of the series $\sum\left(1+\frac{1}{n}\right)^n x^n, x > 0.$

Answer :

Given series is,

$$\sum\left(1+\frac{1}{n}\right)^n x^n,\ x > 0$$

Let,

$$u_n = \left(1+\frac{1}{n}\right)^n .x^n$$

$$\Rightarrow \quad (u_n)^{1/n} = \left[\left(1+\frac{1}{n}\right)^n .x^n\right]^{1/n}$$

$$\Rightarrow \quad (u_n)^{1/n} = \left(1+\frac{1}{n}\right).x$$

$$\operatorname*{Lt}_{n\to\infty} u_n^{1/n} = \operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)x$$

$$= x$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} u_n^{1/n} = x \qquad \text{... (1)}$$

From Cauchy's n^{th} root test, if Σu_n is a series of positive terms such that $\operatorname*{Lt}_{n\to\infty} u_n^{1/n} = p$, then,

(a) Σu_n converges if $p < 1$

(b) Σu_n diverges if $p > 1$ and

(c) Test fails to decide nature if $p = 1$

∴ From equation (1),

Σu_n is convergent if $x < 1$, divergent if $x > 1$ and test fails if $x = 1$

When x = 1

$$u_n = \left(1+\frac{1}{n}\right).x$$

$$\operatorname*{Lt}_{n\to\infty} u_n^{1/n} = \operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)x$$

$$= e \neq 0$$

∴ Σu_n is divergent

Hence the series is convergent if $x < 1$ and divergent if $x \geq 1$.

Q45. Discuss the convergence of the series $\sum\left(\frac{n+2}{n+3}\right)^n x^n$.

Answer :

Given series is,

$$\sum\left(\frac{n+2}{n+3}\right)^n x^n$$

Let,

$$u_n = \sum\left(\frac{n+2}{n+3}\right)^n x^n$$

$$\therefore \quad u_n^{1/n} = \left(\frac{n+2}{n+3}\right)^{\frac{n}{n}} x^{\frac{n}{n}}$$

$$= \left(\frac{n+2}{n+3}\right)x$$

$$= \frac{n}{n}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$\therefore \quad \lim_{n\to\infty} = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$= \left(\frac{1+0}{1+0}\right)x$$

$$= x$$

By Cauchy's n^{th} root test, Σu_n converges if $x < 1$ and diverges if $x > 1$ and fails if $x = 1$

For $x = 1$,

$$u_n = \left(\frac{n+2}{n+3}\right)^n$$

$$= \frac{n^n\left(1+\frac{2}{n}\right)^n}{n^n\left(1+\frac{3}{n}\right)^n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$$

$$= \frac{e^2}{e^3} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2 \quad \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3\right]$$

$$= \frac{1}{e} \neq 0$$

$\therefore$ $\Sigma\, u_n$ is not convergent.

Thus, $\Sigma\, u_n$ is divergent.

Therefore, the series is convergent if $x < 1$ and divergent if $x \geq 1$.

Q46. State and prove Raabe's test.

Answer :

Raabe's Test

If Σu_n is a series of positive terms, and $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = k$. Then,

(i) Series is convergent for $k > 1$

(ii) Series is divergents for $k < 1$

(iii) The test fails for $k = 1$.

Proof

Consider the two series Σu_n and Σv_n

Where, $\Sigma v_n = \Sigma\frac{1}{n^p}$ and it is convergent for $p > 1$.

Case (i) : For k > 1

Assume a number p such that,

$k \geq p < 1$

Comparing Σu_n with Σv_n, which is convergent as $p > 1$,

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$$

$$\text{i.e., } \frac{u_n}{u_{n+1}} \geq \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\geq \left(\frac{n+1}{n}\right)^p$$

$$= \left(1+\frac{1}{n}\right)^p$$

$$= 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

[$\because$ From binomial theorem]

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 \geq \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) \geq p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) \geq \lim_{n\to\infty}\left[p\frac{+(p-1)}{2!}\frac{1}{n}+\ldots\right]$$

If $k > p$ then Σu_n is convergent.

Case (ii) : For k < 1

Select a number p such that,

$k \leq p > 1$

Comparing Σu_n with Σv_n, which is divergent as $p < 1$,

$$\frac{u_n}{u_{n+1}} \leq \frac{v_n}{v_{n+1}}$$

$$\frac{u_n}{u_{n+1}} \leq \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\leq \left(\frac{n+1}{n}\right)^p$$

$$= \left(1+\frac{1}{n}\right)^p$$

$$= 1 + \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots$$

[$\because$ From binomial theorem]

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 \leq \frac{p}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots..$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) \leq p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots..$$

$$\mathop{Lim}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) \leq \mathop{Lim}_{n\to\infty}\left[p + \frac{p(p-1)}{2!}.\frac{1}{n} + \ldots...\right]$$

Therefore, if $k < 1$, then Σu_n is divergent.

Case (iii): For k = 1

If $k = 1$, the test fails because it is not possible to determine whether series is convergent or divergent.

Q47. Discuss the convergence of the series $\sum\frac{\mathbf{4.7.10.... ...(3n+1)}}{\mathbf{1.2.3..... .....n}}\mathbf{x^n}$.

Answer :

Given series is,

$$\Sigma u_n = \sum\frac{4.7.10........(3n+1)}{1.2.3.......n}x^n$$

$$\Rightarrow \quad u_n = \frac{4.7.10.........(3n+1)}{1.2.3.........n}x^n \qquad \ldots (1)$$

$$\Rightarrow \quad u_{n+1} = \frac{4.7.10...........(3n+1)(3n+4)}{1.2.3...........n(n+1)} x^{n+1} \qquad ... (2)$$

Dividing equation (2) by equation (1),

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{4.7.10.......(3n+1)(3n+4)}{1.2.3.............n(n+1)} x^{n+1} \times \frac{1.2.3...........n}{4.7.10.........(3n+1)} . \frac{1}{x^n}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+1)(3n+4)}{(n+1)(3n+1)x^n} x^{n+1}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+4)}{(n+1)} . x$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{n\left(3+\frac{4}{n}\right)}{n\left(1+\frac{1}{n}\right)} x$$

Applying limit on both sides,

$$\Rightarrow \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{\left(3+\frac{4}{n}\right)}{\left(1+\frac{1}{n}\right)} x$$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = 3x$$

(or)

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \frac{1}{3x}$$

By ratio test,

(i) Converges, if $3x < 1$

i.e., $x < \frac{1}{3}$

(ii) Diverges, if $3x > 1$

i.e., $x > \frac{1}{3}$

or

$\frac{1}{3} < x$

(iii) At $x = \frac{1}{3}$, ratio test fails

By applying Raabe's test.

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+4)}{(n+1)} x$$

$$= \frac{(3n+4)}{(n+1)} . \frac{1}{3}$$

$$\Rightarrow \quad \frac{u_{n+1}}{u_n} = \frac{(3n+4)}{(3n+3)}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(3n+3)}{(3n+4)}$$

Substituting '1' on both sides

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{3n+3}{3n+4} - 1$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{3n+3-3n-4}{3n+4}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} - 1 = \frac{-1}{3n+4}$$

Multiplying 'n' on both sides,

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-n}{(3n+4)} = \frac{-n}{n\left(3+\frac{4}{n}\right)}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-1}{3+\frac{4}{n}}$$

$$\underset{n\to\infty}{Lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-1}{3} < 1$$

By Raabe's test the given series is divergent, for $x = \frac{1}{3}$

$\therefore$ The given series is convergent if $x < \frac{1}{3}$ and divergent if $x \geq \frac{1}{3}$.

Q48. Test for convergence $\frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \frac{3.6.9.12}{7.10.13.16}x^4 + ...$

Answer :

Given series is,

$$\frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \frac{3.6.9.12}{7.10.13.16}x^4 + ...$$

Numerator Elements	**Denominator Elements**
3, 6, 9, 12 ...	7, 10, 13, 16 ...
$a = 3$	$a = 7$
$d = 6 - 3 = 3$	$d = 10 - 7 = 3$
$t_n = a + (n-1)d$	$t_n = a + (n-1)d$
$= 3 + (n-1)(3)$	$= 7 + (n-1)(3)$
$= 3n$	$= 3n + 4$

$$\frac{3}{7}x + \frac{3.6}{7.10}x^2 + ... = \sum \frac{3.6...3n}{7.10...(3n+4)} x^n$$

$$u_n = \frac{3.6...3n}{7.10...(3n+4)}x^n \quad ...(1)$$

$$u_{n+1} = \frac{3.6...3n[3(n+1)]x^{n+1}}{7.10...(3n+4)[3(n+1)+4]}$$

$$\Rightarrow \quad u_{n+1} = \frac{3.6...3n(3n+3)]x^n.x^1}{7.10...(3n+4)(3n+7)} \quad ...(2)$$

Dividing equation (2) by equation (1),

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{3.6...(3n)(3n+3)]x^n.x^1}{7.10...(3n+4)(3n+7)} \times \frac{7.10...(3n+4)}{(3.6...3n)x^n}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{(3n+3)x}{3n+7}$$

Applying limit on both sides,

$$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{n\left(3+\frac{3}{n}\right)x}{n\left(3+\frac{7}{n}\right)}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(3+\frac{3}{n}\right)x}{\left(3+\frac{7}{n}\right)}$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = x$$

By Ratio test, $\sum u_n$

(i) Converges if $x < 1$

(ii) Diverges if $x > 1$

(iii) At $x = 1$, ratio test fail.

Applying Raabe's test for $x = 1$,

$$\frac{u_{n+1}}{u_n} = \frac{(3n+3)x}{(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{(3n+3)x}$$

Substituting $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

Substituting '1' on both sides,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{3n+7}{3n+3} - 1$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{3n+7-3n-3}{3n+3}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4}{3n+3}$$

Multiplying 'n' on both sides,

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{4n}{3n+3}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{4n}{n\left[3+\frac{3}{n}\right]}$$

$$\underset{n\to\infty}{\text{Lt}}\, n\left[\frac{u_n}{u_{n+1}} - 1\right] = \underset{n\to\infty}{\text{Lt}} \frac{4}{3+\frac{3}{n}}$$

$$= \frac{4}{3+0}$$

$$= \frac{4}{3} > 1$$

∴ By Raabe's test, $\sum u_n$ is convergent for $x = 1$

∴ The given series converges if $x \le 1$ and diverges if $x > 1$.

Q49. Test for convergence $1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{2.3} + ...$

Answer :

Given series is,

$$1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{2.3} + ...$$

$$\text{Let, } \Sigma u_n = \sum \frac{a(a+1)(a+2)...(a+n)}{1.2.3...n(n+1)}$$

$$\Rightarrow \quad u_n = \frac{a(a+1)...(a+n)}{1.2.3...n(n+1)} \quad ...(1)$$

$$\Rightarrow \quad u_{n+1} = \frac{a(a+1)...(a+n)(a+n+1)}{1.2.3...n(n+1)(n+2)} \quad ...(2)$$

Dividing equation (2) by equation (1)

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{a(a+1)...(a+n)(a+n+1)}{1.2.3...n(n+1)(n+2)} \times \frac{1.2.3...(n+1)}{a(a+1)...(a+n)}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{a+n+1}{n+2}$$

Substracting '1' on both sides,

$$\Rightarrow \frac{u_{n+1}}{u_n} - 1 = \frac{n+2}{a+n+1} - 1$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{n+2-a-n-1}{a+n+1}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{1-a}{a+n+1}$$

Multiplying 'n' on both sides,

$$\Rightarrow \quad n\left[\frac{u_n}{u_{n+1}}-1\right] = \frac{(1-a)n}{a+n+1}$$

Applying limit on both sides,

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ n\left[\frac{u_n}{u_{n+1}}-1\right] = \underset{n\to\infty}{\text{Lt}}\ \frac{(1-a)n}{n\left[\frac{a}{n}+1+\frac{1}{n}\right]}$$

$$= \underset{n\to\infty}{\text{Lt}}\ \frac{(1-a)}{\frac{a}{n}+1+\frac{1}{n}}$$

$$= \frac{1-a}{0+1+0}$$

$$= 1-a$$

By using Raabe's test, $\sum u_n$

(i) Converges if $1-a>1$

$$-a>0$$
$$a<0$$

(ii) Diverges if $1-a<1$

$$-a<0$$
$$a>0$$

Q50. Test the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{1.4.7.....(3n-2)}{2.5.8.......(3n-1)}$.

Answer : **Dec.-16, Q12(a)**

Given series is,

$$\sum_{n=1}^{\infty} \frac{1.4.7.....(3n-2)}{2.5.8.....(3n-1)}$$

Let, $$u_n = \frac{1.4.7.....(3n-2)}{2.5.8.....(3n-1)} \quad ...(1)$$

$$u_{n+1} = \frac{1.4.7.....(3n-2)(3(n+1)-2)}{2.5.8.....(3n-1)(3(n+1)-1)}$$

$$\Rightarrow \quad u_{n+1} = \frac{1.4.7.....(3n-2)(3n+1)}{2.5.8.....(3n-1)(3n+2)} \quad ...(2)$$

Dividing equation (1) by equation (2),

$$\frac{u_n}{u_{n+1}} = \frac{\frac{1.4.7}{2.5.8}\cdots\cdots\frac{(3n-2)}{(3n-1)}}{\frac{1.4.7}{2.5.8}\cdots\cdots\frac{(3n+1)}{(3n+2)}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{3n+2}{3n+1}$$

Substracting '1' on both sides,

$$\Rightarrow \frac{u_n}{u_{n+1}}-1 = \frac{3n+2}{3n+1}-1 = \frac{3n+2-3n-1}{3n+1} = \frac{1}{3n+1}$$

$$\Rightarrow \quad \underset{n\to\infty}{Lim}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = \underset{n\to\infty}{Lim}\ \frac{n}{3n+1}$$

$$= \underset{n\to\infty}{Lim}\, n\frac{n}{n\left(3+\frac{1}{n}\right)} = \frac{1}{3} < 1$$

$\therefore$ By Raabe's test, the series $\sum u_n$ is divergent.

Q51 Text for the convergence of series, $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + ...$

Answer :

Given series is,

$$\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + ...$$

By inspection it is clear that the given series is a series of positive terms.

Consider,

$$u_n = \frac{(a+nx)^n}{n!}$$

$$u_{n+1} = \frac{(a+(n+1)x)^{n+1}}{(n+1)!}$$

Consider,

$$\frac{u_n}{u_{n+1}} = \frac{\frac{(a+nx)^n}{n!}}{\frac{[a+(n+1)x]^{n+1}}{(n+1)!}}$$

Applying limit on both sides,

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lim}}\left[\frac{u_n}{u_{n+1}}\right] = \underset{n\to\infty}{\text{Lim}}\ \frac{\left[\frac{(a+nx)^n}{n!}\right]}{\left[\frac{(a+(n+1)x)^{n+1}}{(n+1)!}\right]}$$

$$= \underset{n\to\infty}{\text{Lim}}\ \frac{(n+1)!(a+nx)^n}{n!(a+(n+1)x)^{n+1}}$$

$$= \underset{n\to\infty}{\text{Lim}}\ \frac{(n+1).n!(a+nx)^n}{n!(a+(n+1)x)^{n+1}}$$

$$= \underset{n\to\infty}{\text{Lim}}\ \frac{(n+1)(a+nx)^n}{[a+(n+1)x]^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)\left[nx\left(\frac{a}{nx}+1\right)\right]^{n}}{\left[(n+1)x\left(\frac{a}{(n+1)x}+1\right)\right]^{n+1}} = \lim_{n\to\infty} \frac{(n+1)n^{n}.x^{n}\left(\frac{a}{nx}+1\right)^{n}}{(n+1)^{n+1}.x^{n+1}\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)n^{n}.x^{n}\left(\frac{a}{nx}+1\right)^{n}}{(n+1)^{n}.(n+1).x^{n+1}\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^{n}.x^{n}\left(\frac{a}{nx}+1\right)^{n}}{(n+1)^{n}\,x^{n}.x\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{n}.\frac{1}{x}.\frac{\left(\frac{a}{nx}+1\right)^{n}}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{n}{n\left(1+\frac{1}{n}\right)}\right)^{n}.\frac{1}{x}.\frac{\left(\frac{a}{nx}+1\right)^{n}}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right] = \lim_{n\to\infty} \left[\left(\frac{1}{\left(1+\frac{1}{n}\right)^{n}}.\frac{1}{x}.\frac{\left(\frac{a}{nx}+1\right)^{n}}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{\left[1+\left(\frac{1}{n}\right)^{n}\right]}\right] \times \lim_{n\to\infty}\left(\frac{1}{x}\right) \times \lim_{n\to\infty}\left[\frac{\left(\frac{a}{nx}+1\right)^{n}}{\left(\frac{a}{(n+1)x}+1\right)^{n+1}}\right]$$

$$= \frac{1}{e} \times \frac{1}{x} \times \lim_{n\to\infty}\left[\frac{\left(\frac{a}{nx}+1\right)^{n}}{\left(\frac{a}{(n+1)x}+1\right)}\right] \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e\right]$$

$$= \frac{1}{e} \times \frac{1}{x} \times \frac{\lim_{n\to\infty}\left(\frac{a}{nx}+1\right)^{n}}{\lim_{n\to\infty}\left(\frac{a}{(n+1)x}+1\right)^{n+1}}$$

$$= \frac{1}{e.x} \times \frac{e^{a/x}}{e^{a/x}} \qquad \left[\because \lim_{n\to\infty}\left(\frac{a}{nx}+1\right)^{n} = e^{a/x}\right]$$

$$= \frac{1}{e.x}$$

By ratio test, the given series is convergent if $\frac{1}{e.x} > 1$ i.e., $x < \frac{1}{e}$ and the series is divergent if $\frac{1}{e.x} < 1$ i.e., $x > \frac{1}{e}$. If $x = \frac{1}{e}$ then test fails.

When $x = \frac{1}{e}$,

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{1}{\left(\frac{1}{e}\right)} \times \frac{\left(\frac{a}{n\left(\frac{1}{e}\right)}+1\right)^n}{\left(\frac{a}{(n+1)\left(\frac{1}{e}\right)}+1\right)^{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n} \times \frac{\left(\frac{ae}{n}+1\right)^n}{\left(\frac{ae}{(n+1)}+1\right)^{n+1}}$$

Since $\frac{u_n}{u_{n+1}}$ involves 'e', we apply logarithmic test.

Applying log on both sides,

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log\left[\frac{e.\left(\frac{ae}{n}+1\right)^n}{\left(1+\frac{1}{n}\right)^n\left(\frac{ae}{(n+1)}+1\right)^{n+1}}\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log e + \log\left(\frac{ae}{n}+1\right)^n - \log\left(1+\frac{1}{n}\right)^n - \log\left(\frac{ae}{(n+1)}+1\right)^{n+1}$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = 1 + n\log\left(\frac{ae}{n}+1\right) - n\log\left(1+\frac{1}{n}\right) - (n+1)\log\left(\frac{ae}{(n+1)}+1\right)$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = 1 + n\left(\frac{ae}{n} - \frac{a^2e^2}{2n^2} + \frac{a^3e^3}{3n^3} \ldots\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \ldots\right) - (n+1)\left(\frac{ae}{(n+1)} - \frac{a^2e^2}{2(n+1)^2} + \frac{a^3e^3}{3(n+1)^3} \ldots\right)$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = 1 + ae - \frac{a^2e^2}{2n} + \frac{a^3e^3}{3n^2} - \ldots - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \ldots - ae + \frac{a^2e^2}{2(n+1)} - \frac{a^3e^3}{3(n+1)^2} + \ldots$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-a^2e^2}{2n} + \frac{a^3e^3}{3n^2} + \frac{1}{2n} - \frac{1}{3n^2} + \frac{a^2e^2}{2(n+1)} - \frac{a^3e^3}{3(n+1)^2} + \ldots$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1-a^2e^2}{2n} + \frac{a^2e^2}{2(n+1)} + \ldots$$

Multiplying 'n' on both sides,

$$\Rightarrow \quad n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1-a^2e^2}{2} + \frac{n.a^2e^2}{2(n+1)} + \ldots$$

Applying limit on both sides,

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \lim_{n\to\infty}\left[\frac{1-a^2e^2}{2}+\frac{na^2e^2}{2(n+1)}+...\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \lim_{n\to\infty}\frac{1-a^2e^2}{2} + \lim_{n\to\infty}\frac{na^2e^2}{2(n+1)} + ...$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2}(1) + \lim_{n\to\infty}\frac{na^2e^2}{2n\left(1+\frac{1}{n}\right)} + ... \qquad \left[\because \lim_{n\to\infty}(1)=1\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2} + \lim_{n\to\infty}\frac{a^2e^2}{2\left(1+\frac{1}{n}\right)} + ...$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2} + \frac{a^2e^2}{2\left(1+\frac{1}{\infty}\right)} + ...$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1-a^2e^2}{2} + \frac{a^2e^2}{2(1+0)} + ...$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \left(\frac{1}{2}-\frac{a^2e^2}{2}\right) + \frac{a^2e^2}{2(1)} + ...$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1}{2} - \frac{a^2e^2}{2} + \frac{a^2e^2}{2} + ...$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{1}{2} < 1$$

Since, the value obtained is less than one, by logarithmic test the given series is divergent.

Thus the series converges if $x < \frac{1}{e}$ and diverges if $x \geq \frac{1}{e}$.

Q52. Test the following series for the convergence,

$$1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + ...$$

Answer :

Given series is,

$$1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + ...$$

From the given series,

$$u_n = \frac{n!x^n}{(n+1)^n}$$

$$u_{n+1} = \frac{(n+1)!x^{n+1}}{(n+2)^{n+1}}$$

Consider,

$$\frac{u_n}{u_{n+1}} = \frac{\left[\dfrac{n!x^n}{(n+1)^n}\right]}{\left[\dfrac{(n+1)!x^{n+1}}{(n+2)^{n+1}}\right]}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{n!.x^n.(n+2)^{n+1}}{(n+1)^n.(n+1)!x^{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{n!.x^n.(n+2)^{n+1}}{(n+1)^n.(n+1)n!x^n.x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(n+2)^{n+1}}{(n+1).(n+1)^n.x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\frac{(n+2)}{(n+1)}\right]^{n+1}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\frac{(n+1)+1}{(n+1)}\right]^{n+1}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\left(\frac{n+1}{n+1}+\frac{1}{(n+1)}\right)^{n+1}\right].\frac{1}{x}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} = \left[\left(1+\frac{1}{n+1}\right)^{n+1}\right].\frac{1}{x}$$

Applying limit on both sides,

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left\{\left[1+\frac{1}{(n+1)}\right]^{n+1}.\frac{1}{x}\right\}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{x}\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{x}.e \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{e}{x}$$

By ratio test, it is clear that the series is convergent if $\frac{e}{x} > 1$ i.e., $x < e$ and diverges if $\frac{e}{x} < 1$ i.e., $x > e$ and the test fails when $\frac{e}{x} = 1 \Rightarrow x = e$.

When $x = e$,

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n+1}\right)^{n+1}.\frac{1}{e}$$

since $\frac{u_n}{u_{n+1}}$ involves e we apply logarithmic test. Applying log on both sides,

$$\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left[\frac{1}{e}\left(1+\frac{1}{(n+1)}\right)^{n+1}\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log\frac{1}{e} + \log\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \log e^{-1} + (n+1)\log\left(1+\frac{1}{n+1}\right)$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = -\log e + (n+1)\left(\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots\right) \quad \left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = -1 + (n+1)\left[\frac{1}{(n+1)} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \ldots\right]$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = -1 + 1 - \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} + \ldots$$

$$\Rightarrow \quad \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-1}{2(n+1)} + \frac{1}{3(n+1)^2} + \ldots$$

Multiply 'n' on both sides,

$$\Rightarrow n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-n}{2(n+1)} + \frac{n}{3(n+1)^2} + \ldots$$

$$\Rightarrow n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-n}{2n\left(1+\frac{1}{n}\right)} + \frac{n}{3n^2\left(1+\frac{1}{n}\right)^2} + \ldots$$

$$\Rightarrow \quad n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{-1}{2\left(1+\frac{1}{n}\right)} + \frac{1}{3n\left(1+\frac{1}{n}\right)^2}$$

Applying limit on both sides,

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \underset{n\to\infty}{Lt}\left[\frac{-1}{2\left(1+\frac{1}{n}\right)} + \frac{1}{3n\left(1+\frac{1}{n}\right)^2} + \ldots\right]$$

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{-1}{2\left(1+\frac{1}{\infty}\right)} + \frac{1}{3\infty\left(1+\frac{1}{\infty}\right)^2} + \ldots$$

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{-1}{2(1+0)}+0...$$

$$\Rightarrow \underset{n\to\infty}{Lt}\left[n\log\left(\frac{u_n}{u_{n+1}}\right)\right] = \frac{-1}{2(1)} = \frac{-1}{2}<1$$

∴ The series is divergent series.

Thus, the series converges if $x < e$ and diverges if $x \geq e$.

1.4 ALTERNATING SERIES, SERIES OF POSITIVE AND NEGATIVE TERMS

Q53. State and prove Leibnitz's test.

Answer :

If $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is an alternating series then it is convergent if,

(a) $u_1 \geq u_2 > u_3 \geq ... \geq u_n \geq u_{n+1}$ ....

(b) $\underset{n\to\infty}{Lt}\, u_n = 0$

Proof

Consider $s_n = u_1 - u_2 + u_3 - u_4 + ... (-1)^{n-1} u_n$

And, $s_{2n} = u_1 - u_2 + u_3 - u_4 + ... + u_{2n-1} - u_{2n}$ ... (1)

$s_{2n+2} = u_1 - u_2 + u_3 - u_4 + ...$ ... (2)

$u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$ ... (3)

By substracting equation (3) from equation (2),

$s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geq 0$ [∵ From condition (a)]

∴ $s_{2n+2} \geq s_{2n}\ \forall\ n$

The subsequence $\{s_{2n}\}$ of $\{s_n\}$ is an increasing sequence.

From equation (2),

$s_{2n} = u_1 - [(u_2 - u_3) + ... + (u_{2n-2} - u_{2n-1}) + u_{2n}]$

$= u_1$ – a positive number [∵ $u_2 – u_3$, ... $u_{2n-2} - u_{2n-1}, u_{2n}$ are positive]

∴ $s_{2n} \leq u_1\ \forall\ n$

∴ $\{s_{2n}\}$ is bounded above

∴ $\{s_{2n}\}$ converges.

Assume $\lim_{n\to\infty} s_{2n = l}$ ... (4)

∵ $s_{2n-1} = s_{2n} + u_{2n}$

∴ $\lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} u_{2n}$

$= l + 0$

[∵ From equation (4) and condition (b)]

∴ $\{s_{2n-1}\}$ also converges to l

∴ $\{s_n\}$ converges to l

Hence, an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ also convergent.

Q54. Discuss the convergence of the series $\sum \frac{\cos n\pi}{n^2+1}$.

Answer :

Given that,

$$\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$= \frac{-1}{2}+\frac{1}{2^2+1}-\frac{1}{3^2+1}+.....+\frac{(-1)^n}{n^2+1}$$

$$u_1 = \frac{1}{2},\ u_2 = \frac{1}{5},\ u_3 = \frac{1}{10},\ u_4 = \frac{1}{17},\ ...,\ u_n = \frac{1}{n^2+1}$$

$u_1 > u_2 > u_3 > ...\ u_{n-1} > u_n$

$\Rightarrow \quad u_{n-1} > u_n$

$\Rightarrow \quad u_n < u_{n-1}$

And $\underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\, \frac{1}{n^2+1} = 0$

$\therefore$ From Leibnitz's test the given series is convergent.

$\therefore \quad \sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2+1}$ is convergent.

Q55. Discuss the convergence of the series

$$\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+...\ (0 < x < 1)$$

Answer :

Given that,

$$\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}...... = \sum\frac{x^n}{1+x^n}(-1)^{n-1}$$

Let, $u_n = \frac{x^n}{1+x^n}$

Since,

$1 + x^{n-1} < 1 + x^n$

$\Rightarrow \quad \frac{1}{1+x^{n-1}} > \frac{1}{1+x^n}$

$\Rightarrow \quad \frac{x^{n-1}}{1+x^{n-1}} > \frac{x^n}{1+x^n}$

$\Rightarrow \quad u_{n-1} > u_n$

$\Rightarrow \quad u_n < u_{n-1}$

$$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{x^n}{1+x^n} = \underset{n\to\infty}{\text{Lt}}\, \frac{x^n}{x^n\left(\frac{1}{x^n}+1\right)} = \frac{1}{\infty+1} = 0$$

From Leibnitz test the given series is convergent.

$\therefore \quad \sum\frac{x^n}{1+x^n}(-1)^{n-1}$ is convergent.

Q56. Examine the convergence of the alternating series $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}$ ...

OR

By Leibnitz's test, verify the series $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\frac{1}{7.8}$ ... is convergent.

Answer :

Given series is,

$$\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-...$$

First Elements	**Second Elements**
1, 3, 5 ...	2, 4, 6, 8 ...
$a = 1$,	$a = 4$
$d = 3 - 1 = 2$	$d = 4 - 2 = 2$
$t_n = a + (n-1)\,d$ $= 1 + (n-1)(2)$ $= 2n - 1$	$t_n = a + (n-1)\,d = 2 + (n-1)(2) = 2n$

$$\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}\ ... = \Sigma\,(-1)^{n-1}\frac{1}{(2n-1)(2n)}$$

Let,

$$u_n = \frac{1}{(2n-1)(2n)}$$

$$u_{n-1} = \frac{1}{(2n-3)(2n-2)}$$

Consider,

$$u_n - u_{n-1} = \frac{1}{(2n-1)(2n)} - \frac{1}{(2n-3)(2n-2)}$$

$$= \frac{(2n-3)(2n-2)-(2n-1)(2n)}{(2n)(2n-1)(2n-2)(2n-3)}$$

$$< 0$$

$$\therefore \quad u_n - u_{n-1} < 0$$

$$\therefore \quad u_n < u_{n-1}$$

$$\underset{n\to\infty}{Lim}\ u_n = \underset{n\to\infty}{Lim}\ \frac{1}{(2n-1)(2n)} = \frac{1}{\infty} = 0$$

From Leibnitz test $\Sigma\,(-1)^{n-1}\frac{1}{(2n-1)(2n)}$ is convergent.

Q57. State Leibnitz test and use it to test the convergence of the series $\sum(-1)^n\frac{n}{2n+1}$.

Answer :

Leibnitz Test

For answer refer Unit-1, Q14.

Given series is,

$$\sum(-1)^n\frac{n}{2n+1}$$

The terms of the series are alternatively positive and negative.

Let,

$$u_n = \frac{n}{2n+1}$$

$$\Rightarrow \quad u_{n+1} = \frac{n}{2(n+1)+1} = \frac{n+1}{2n+3}$$

$$u_n - u_{n+1} = \frac{n}{2n+1} - \frac{n+1}{2n+3}$$

$$= \frac{2n^2+3n-2n^2-2n-n-1}{(2n+1)(2n+3)}$$

$$= \frac{-1}{(2n+1)(2n+3)}$$

$$\therefore \quad u_n < u_{n+1}$$

$$\underset{n\to\infty}{Lim}\, u_n = \underset{n\to\infty}{Lim} \frac{n}{2n+1} = \underset{n\to\infty}{Lim} \frac{n}{n(2+1/n)}$$

$$= \underset{n\to\infty}{Lim} \frac{1}{2+1/n} = \frac{1}{2} \neq 0$$

Both conditions of convergence are not satisfied.

$\therefore$ By Leibnitz test, the series is not convergent. It is oscillatory.

1.5 ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

Q58. Test the series $\Sigma(-1)^{n-1}\frac{1}{n2^n}$ for absolute convergence.

Answer :

Given series,

$$\Sigma(-1)^{n-1}\frac{1}{n2^n}$$

Let, $u_n = (-1)^{n-1}\frac{1}{n2^n}$

$$\Rightarrow \quad u_{n+1} = (-1)^{n+1-1}\frac{1}{(n+1)2^{n+1}} = \frac{(-1)^n}{(n+1)2^{n+1}}$$

Consider,

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{(-1)^n}{(n+1)2^{n+1}}}{\dfrac{(-1)^{n-1}}{n.2^n}}$$

$$= \frac{(-1)^n}{(n+1)2^{n+1}} \times \frac{n.2^n}{(-1)^{n-1}}$$

$$= \frac{(-1)^n}{(n+1).2^n.2^1} \times \frac{n.2^n}{(-1)^n.(-1)^{-1}}$$

$$= \frac{n}{2(n+1).(-1)^{-1}} = \frac{n(-1)}{2(n+1)}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{-n}{2(n+1)} \qquad \left[\because \frac{1}{(-1)^{-1}} = -1\right]$$

Then, $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{-n}{2(n+1)}\right|$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}} \frac{n}{2(n+1)}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{n}{2n\left(1+\frac{1}{n}\right)}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{2\left(1+\frac{1}{n}\right)} = \frac{1}{2\left(1+\frac{1}{\infty}\right)}$$

$$= \frac{1}{2(1+0)} = \frac{1}{2} < 1$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}} \frac{|u_{(n+1)}|}{|u_n|} < 1$$

$\therefore$ Σu_n is absolute convergent.

Q59. Test the following series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ for conditional convergence.

Answer : Dec.-17, Q12(b)

Given series is,

$$\sum(-1)^{n-1}\frac{n}{n^2+1}$$

Let,

$$u_n = (-1)^{n-1}.\frac{n}{n^2+1}$$

$$\Rightarrow \quad u_n = (-1)^{n-1}.\, v_n$$

Where,

$$v_n = \frac{n}{n^2+1}$$

Σu_n is an a alternating series.

i.e., It contains terms which are alternatively positive and negative.

Consider,

$$v_n = \frac{n}{n^2+1}$$

$$\Rightarrow \quad v_n > 0 \;\forall\; n$$

$$v_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$\therefore \quad v_n > v_{n+1} \; \forall \; n$$

Also, $\underset{n\to\infty}{Lt}\, v_n = \underset{n\to\infty}{Lt} \frac{n}{n^2+1}$

$$= \underset{n\to\infty}{Lt} \frac{n}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{n}\left(1+\frac{1}{n^2}\right)$$

$$= 0$$

Since, $v_n > 0$, $v_n > v_{n+1}$ and $\underset{n\to\infty}{Lt}\, v_n = 0$

Thus, by Leibnitz's test the alternating series is a convergent series.

$\therefore \quad \Sigma u_n$ is convergent

Test for Absolute Convergence or Conditional Convergence

$$u_n = (-1)^{n-1} \cdot \frac{n}{n^2+1}$$

$$\Rightarrow \quad |u_n| = \left|(-1)^{n-1} \cdot \frac{n}{n^2+1}\right|$$

$$\Rightarrow \quad |u_n| = \frac{n}{n^2+1}$$

Let,

$$\Sigma v_n = \sum \frac{1}{n}$$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{|u_n|}{v_n} = \underset{n\to\infty}{Lt} \frac{n}{n^2+1} \cdot \frac{1}{\frac{1}{n}} = \underset{n\to\infty}{Lt} \frac{n^2}{n^2+1}$$

$$= \underset{n\to\infty}{Lt} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = \underset{n\to\infty}{Lt} \frac{1}{1+\frac{1}{n^2}}$$

$$= 1 \neq 0$$

$\therefore$ By comparison test, Σv_n is divergent

Hence, $\Sigma|u_n|$ is also divergent.

$\therefore$ Σv_n is convergent but not absolutely convergent i.e., it converges conditionally.

Q60. Prove that the series $\sum (-1)^{n-1} \frac{\sin nx}{n^2}$ converges absolutely.

Answer : June/July-17, Q12(b)

Given series is,

$$\Sigma(-1)^{n-1} \frac{\sin nx}{n^2}$$

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\therefore\ \Sigma(-1)^{n-1}\dfrac{\sin nx}{n^2}$ is convergent if $\Sigma\left|(-1)^{n-1}\dfrac{\sin nx}{n^2}\right|$ converges.

$$\left|\frac{\sin nx}{n^2}\right| \le \frac{1}{n^2} \qquad [\because |\sin nx| \le 1]$$

$\Sigma\dfrac{1}{n^2}$ is a power series with power = 2 which is greater than 1.

$\therefore\quad \Sigma\dfrac{1}{n^2}$ is convergent, and hence $\Sigma(-1)^{n-1}\dfrac{\sin nx}{n^2}$ is absolutely convergent.

Q61. Examine whether the series

$-1+\dfrac{1}{2^2}-\dfrac{1}{3^2}+\dfrac{1}{4^2}-\dfrac{1}{5^2}+....$ is absolutely convergent or conditionally convergent.

Answer : Dec.-16, Q12(b)

Given series is,

$$-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+.....$$

$$\Rightarrow\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+.....$$

Let,

$$\sum_{n=1}^{\infty}u_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty}(-1)\,v_n$$

Where, $v_n = \dfrac{1}{n^2} > 0\ \forall n$

$$v_{n+1} = \frac{1}{(n+1)^2}$$

Since $n^2 < (n+1)^2$;

$\Rightarrow\quad v_n > v_{n+1}\ \forall n$

$$\Rightarrow\quad \underset{n\to\infty}{lt}\ v_n = \underset{n\to\infty}{lt}\ \frac{1}{n^2} = 0$$

$\therefore\quad$ By Leibnitz's test, the given series is convergent

As $u_n = (-1)^n\dfrac{1}{n^2}$;

$|u_n| = \dfrac{1}{n^2}$;

By p-test, $|u_n| = \dfrac{1}{n^2}$ is convergent with $p = 2$.

As $|u_n|$ is convergent, Σu_n is absolutely convergent.

Hence, the given series $-1+\dfrac{1}{2^2}-\dfrac{1}{3^2}+\dfrac{1}{4^2}-\dfrac{1}{5^2}+....$ is absolutely convergent.