Chapter 1

Introduction

In this first chapter, we introduce the ideas behind optimization and optimal control and provide a brief history of calculus of variations and optimal control. Also, a brief summary of chapter contents is presented.

1.1 Classical and Modern Control

The *classical* (conventional) control theory concerned with single input and single output (SISO) is mainly based on Laplace transforms theory and its use in system representation in block diagram form. From Figure 1.1, we see that

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{1.1.1}$$

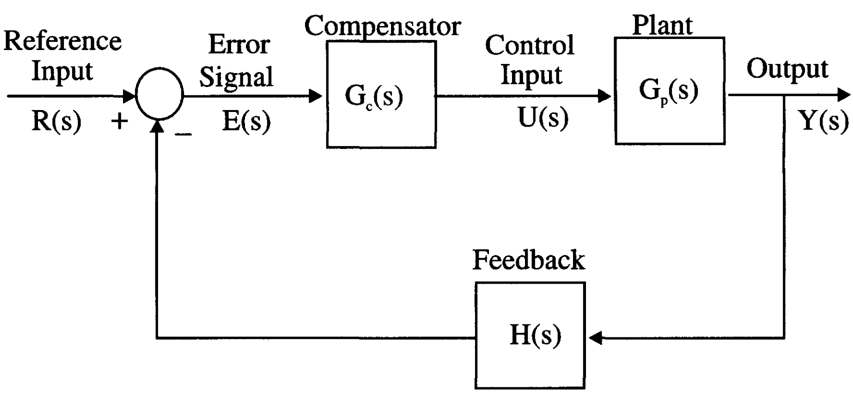

Figure 1.1 Classical Control Configuration

where s is Laplace variable and we used

$$G(s) = G_c(s)G_p(s). \tag{1.1.2}$$

Note that

1. the input $u(t)$ to the plant is determined by the error $e(t)$ and the compensator, and

2. all the variables are not readily available for feedback. In most cases only one output variable is available for feedback.

The *modern* control theory concerned with multiple inputs and multiple outputs (MIMO) is based on state variable representation in terms of a set of first order differential (or difference) equations. Here, the system (plant) is characterized by state variables, say, in *linear*, time-invariant form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1.1.3}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{1.1.4}$$

where, *dot* denotes differentiation with respect to (w.r.t.) t, $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\mathbf{y}(t)$ are n, r, and m dimensional *state*, *control*, and *output* vectors respectively, and A is nxn state, B is nxr input, C is mxn output, and D is mxr transfer matrices. Similarly, a *nonlinear* system is characterized by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.1.5}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t). \tag{1.1.6}$$

The modern theory dictates that all the state variables should be fed back after suitable weighting. We see from Figure 1.2 that in modern control configuration,

1. the input $\mathbf{u}(t)$ is determined by the controller (consisting of error detector and compensator) driven by system states $\mathbf{x}(t)$ and reference signal $\mathbf{r}(t)$,

2. all or most of the state variables are available for control, and

3. it depends on well-established matrix theory, which is amenable for large scale computer simulation.

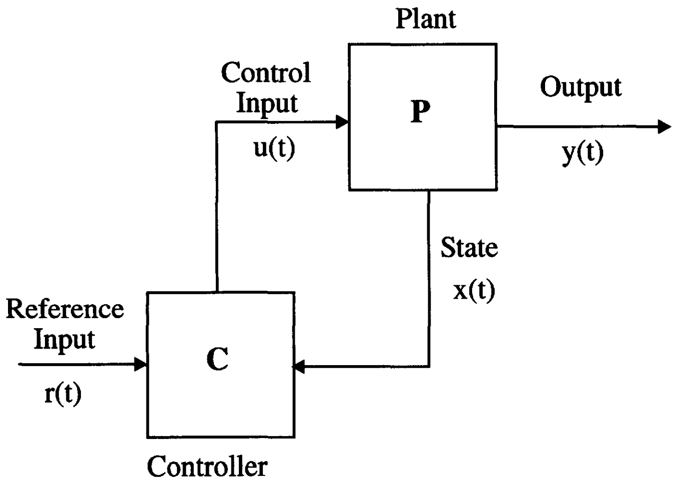

Figure 1.2 Modern Control Configuration

The fact that the state variable representation *uniquely* specifies the transfer function while there are a number of state variable representations for a given transfer function, reveals the fact that state variable representation is a more complete description of a system.

Figure 1.3 shows components of a modern control system. It shows three components of modern control and their important contributors. The first stage of any control system theory is to obtain or formulate the dynamics or *modeling* in terms of dynamical equations such as differential or difference equations. The system dynamics is largely based on the Lagrangian function. Next, the system is *analyzed* for its performance to find out mainly stability of the system and the contributions of Lyapunov to stability theory are well known. Finally, if the system performance is not according to our specifications, we resort to *design* [25, 109]. In optimal control theory, the design is usually with respect to a performance index. We notice that although the concepts such as Lagrange function [85] and V function of Lyapunov [94] are *old*, the techniques using those concepts are *modern*. Again, as the phrase *modern* usually refers to time and what is modern today becomes ancient after a few years, a more appropriate thing is to label them as optimal control, nonlinear control, adaptive control, robust control and so on.

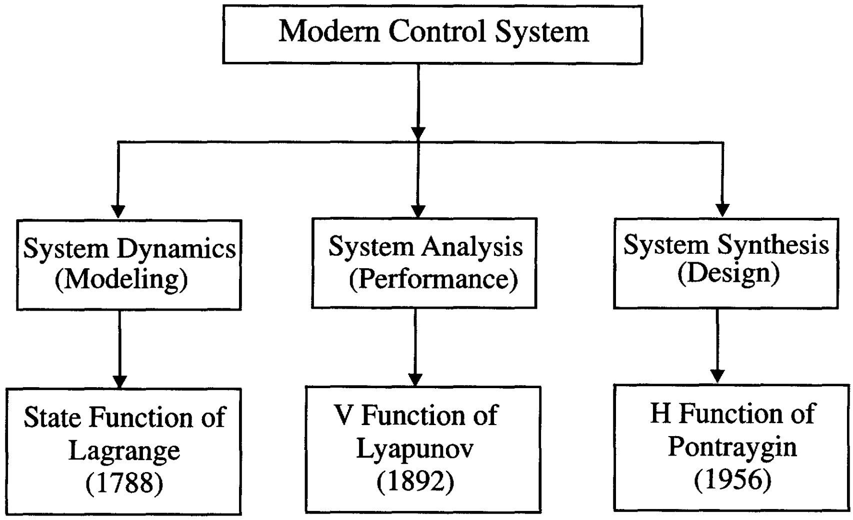

Figure 1.3 Components of a Modern Control System

1.2 Optimization

Optimization is a very desirable feature in day-to-day life. We like to work and use our time in an optimum manner, use resources optimally and so on. The subject of optimization is quite general in the sense that it can be viewed in different ways depending on the *approach* (algebraic or geometric), the *interest* (single or multiple), the *nature* of the signals (deterministic or stochastic), and the *stage* (single or multiple) used in optimization. This is shown in Figure 1.4. As we notice that the calculus of variations is one small area of the big picture of the optimization field, and it forms the basis for our study of optimal control systems. Further, optimization can be classified as *static* optimization and *dynamic* optimization.

1. **Static Optimization** is concerned with controlling a plant under *steady state* conditions, i.e., the system variables are not changing with respect to time. The plant is then described by *algebraic* equations. Techniques used are ordinary calculus, Lagrange multipliers, linear and nonlinear programming.

2. **Dynamic Optimization** concerns with the optimal control of plants under *dynamic* conditions, i.e., the system variables are changing with respect to time and thus the time is involved in system description. Then the plant is described by *differential*

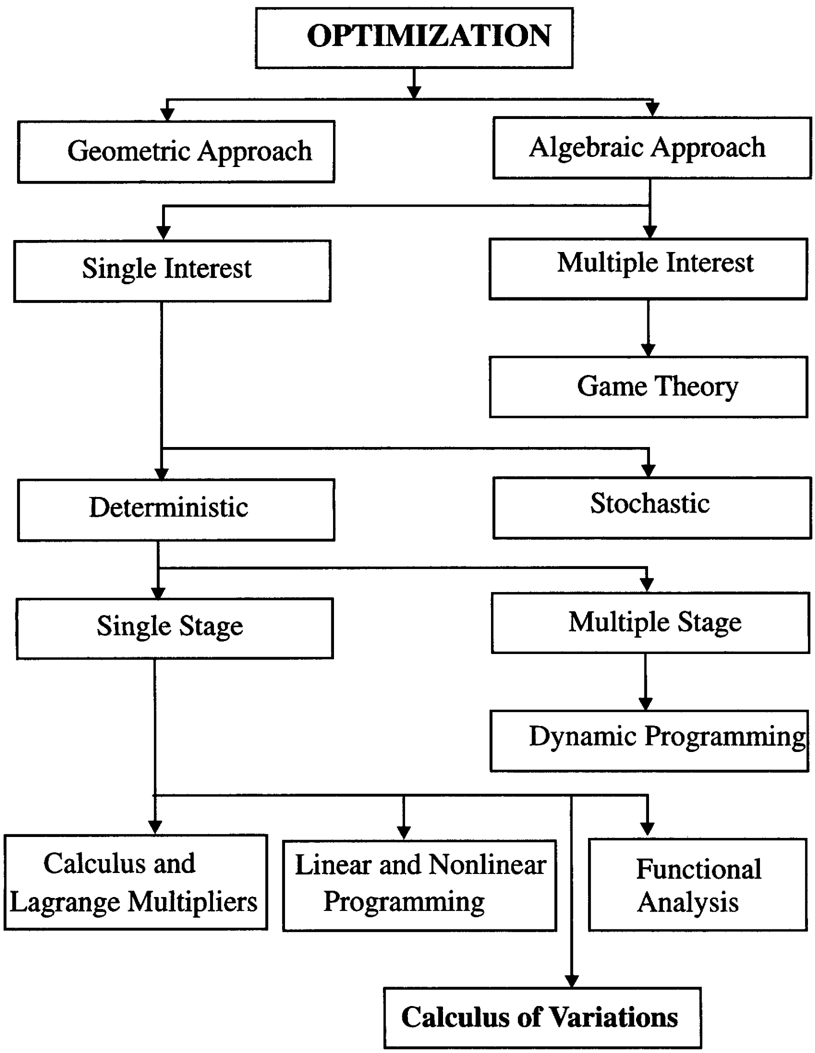

Figure 1.4 Overview of Optimization

(or difference) equations. Techniques used are search techniques, dynamic programming, variational calculus (or calculus of variations) and Pontryagin principle.

1.3 Optimal Control

The main objective of optimal control is to determine control signals that will cause a process (plant) to satisfy some physical constraints and at the same time extremize (maximize or minimize) a chosen performance criterion (performance index or cost function). Referring to Figure 1.2, we are interested in finding the optimal control $\mathbf{u}^*(t)$ (* indicates optimal condition) that will drive the plant P from initial state to final state with some constraints on controls and states and at the same time extremizing the given performance index J.

The formulation of optimal control problem requires

1. a mathematical description (or model) of the process to be controlled (generally in state variable form),
2. a specification of the performance index, and
3. a statement of boundary conditions and the physical constraints on the states and/or controls.

1.3.1 Plant

For the purpose of optimization, we describe a physical plant by a set of linear or nonlinear differential or difference equations. For example, a linear time-invariant system is described by the state and output relations (1.1.3) and (1.1.4) and a nonlinear system by (1.1.5) and (1.1.6).

1.3.2 Performance Index

Classical control design techniques have been successfully applied to linear, time-invariant, single-input, single output (SISO) systems. Typical performance criteria are system time response to step or ramp input characterized by rise time, settling time, peak overshoot, and steady state accuracy; and the frequency response of the system characterized by gain and phase margins, and bandwidth.

In modern control theory, the optimal control problem is to find a control which causes the dynamical system to reach a target or follow a state variable (or trajectory) and at the same time extremize a performance index which may take several forms as described below.

1. **Performance Index for Time-Optimal Control System:** We try to transfer a system from an arbitrary initial state $\mathbf{x}(t_0)$ to a specified final state $\mathbf{x}(t_f)$ in *minimum* time. The corresponding performance index (PI) is

$$J = \int_{t_0}^{t_f} dt = t_f - t_0 = t^*. \tag{1.3.1}$$

2. **Performance Index for Fuel-Optimal Control System:** Consider a spacecraft problem. Let $u(t)$ be the thrust of a rocket engine and assume that the magnitude $|u(t)|$ of the thrust is proportional to the *rate* of fuel consumption. In order to *minimize* the total expenditure of fuel, we may formulate the performance index as

$$J = \int_{t_0}^{t_f} |u(t)| dt. \tag{1.3.2}$$

   For several controls, we may write it as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} R_i |u_i(t)| dt \tag{1.3.3}$$

   where R is a weighting factor.

3. **Performance Index for Minimum-Energy Control System:** Consider $u_i(t)$ as the current in the ith loop of an electric network. Then $\sum_{i=1}^{m} u_i^2(t) r_i$ (where, r_i is the resistance of the ith loop) is the total power or the total *rate* of energy expenditure of the network. Then, for minimization of the total expended energy, we have a performance criterion as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2(t) r_i dt \tag{1.3.4}$$

   or in general,

$$J = \int_{t_0}^{t_f} \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t) dt \tag{1.3.5}$$

where, $\mathbf{R}$ is a *positive definite* matrix and prime ($'$) denotes transpose here and throughout this book (see Appendix A for more details on *definite* matrices).

Similarly, we can think of minimization of the integral of the squared error of a tracking system. We then have,

$$J = \int_{t_0}^{t_f} \mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t)dt \tag{1.3.6}$$

where, $\mathbf{x}_d(t)$ is the desired value, $\mathbf{x}_a(t)$ is the actual value, and $\mathbf{x}(t) = \mathbf{x}_a(t) - \mathbf{x}_d(t)$, is the error. Here, $\mathbf{Q}$ is a weighting matrix, which can be *positive semi-definite.*

4. **Performance Index for Terminal Control System:** In a terminal target problem, we are interested in minimizing the error between the desired target position $\mathbf{x}_d(t_f)$ and the actual target position $\mathbf{x}_a(t_f)$ at the end of the maneuver or at the final time t_f. The terminal (final) error is $\mathbf{x}(t_f) = \mathbf{x}_a(t_f) - \mathbf{x}_d(t_f)$. Taking care of positive and negative values of error and weighting factors, we structure the cost function as

$$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) \tag{1.3.7}$$

which is also called the *terminal cost function.* Here, $\mathbf{F}$ is a *positive semi-definite matrix.*

5. **Performance Index for General Optimal Control System:** Combining the above formulations, we have a performance index in general form as

$$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]dt \tag{1.3.8}$$

or,

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{1.3.9}$$

where, $\mathbf{R}$ is a positive definite matrix, and $\mathbf{Q}$ and $\mathbf{F}$ are positive semidefinite matrices, respectively. Note that the matrices $\mathbf{Q}$ and $\mathbf{R}$ may be time varying. The particular form of performance index (1.3.8) is called *quadratic* (in terms of the states and controls) form.

The problems arising in optimal control are classified based on the structure of the performance index J [67]. If the PI (1.3.9) contains the *terminal* cost function $S(\mathbf{x}(t), \mathbf{u}(t), t)$ *only*, it is called the *Mayer* problem, if the PI (1.3.9) has *only* the *integral* cost term, it is called the *Lagrange* problem, and the problem is of the *Bolza* type if the PI contains both the *terminal* cost term and the *integral* cost term as in (1.3.9). There are many other forms of cost functions depending on our performance specifications. However, the above mentioned performance indices (with quadratic forms) lead to some very elegant results in optimal control systems.

1.3.3 Constraints

The control $\mathbf{u}(t)$ and state $\mathbf{x}(t)$ vectors are either *unconstrained* or *constrained* depending upon the physical situation. The unconstrained problem is less involved and gives rise to some elegant results. From the physical considerations, often we have the controls and states, such as currents and voltages in an electrical circuit, speed of a motor, thrust of a rocket, constrained as

$$\mathbf{U}_{+} \leq \mathbf{u}(t) \leq \mathbf{U}_{-}, \quad \text{and} \quad \mathbf{X}_{-} \leq \mathbf{x}(t) \leq \mathbf{X}_{+} \tag{1.3.10}$$

where, $+$, and $-$ indicate the maximum and minimum values the variables can attain.

1.3.4 Formal Statement of Optimal Control System

Let us now state formally the optimal control problem even risking repetition of some of the previous equations. The optimal control problem is to find the optimal control $\mathbf{u}^*(t)$ ($*$ indicates extremal or optimal value) which causes the *linear* time-invariant plant (system)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1.3.11}$$

to give the trajectory $\mathbf{x}^*(t)$ that optimizes or extremizes (minimizes or maximizes) a performance index

$$J = \mathbf{x}'(t_f)\mathbf{F}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]dt \tag{1.3.12}$$

or which causes the *nonlinear* system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.3.13}$$

to give the state $\mathbf{x}^*(t)$ that optimizes the general performance index

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{1.3.14}$$

with some constraints on the control variables $\mathbf{u}(t)$ and/or the state variables $\mathbf{x}(t)$ given by (1.3.10). The final time t_f may be *fixed*, or *free*, and the final (target) state may be *fully* or *partially fixed* or *free*. The entire problem statement is also shown pictorially in Figure 1.5. Thus,

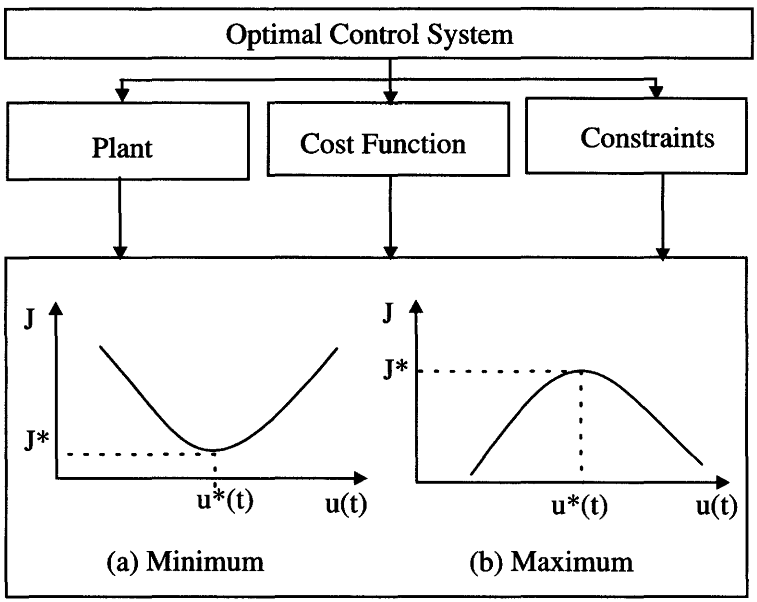

Figure 1.5 Optimal Control Problem

we are basically interested in finding the control $\mathbf{u}^*(t)$ which when applied to the plant described by (1.3.11) or (1.3.13), gives an optimal performance index J^* described by (1.3.12) or (1.3.14).

The optimal control systems are studied in three stages.

1. In the first stage, we just consider the performance index of the form (1.3.14) and use the well-known theory of calculus of variations to obtain optimal functions.

2. In the second stage, we bring in the plant (1.3.11) and try to address the problem of finding optimal control $\mathbf{u}^*(t)$ which will

drive the plant and at the same time optimize the performance index (1.3.12). Next, the above topics are presented in discrete-time domain.

3. Finally, the topic of constraints on the controls and states (1.3.10) is considered along with the plant and performance index to obtain optimal control.

1.4 Historical Tour

We basically consider two stages of the tour: first the development of calculus of variations, and secondly, optimal control theory [134, 58, 99, 28][1].

1.4.1 Calculus of Variations

According to a legend [88], Tyrian princess Dido used a rope made of cowhide in the form of a circular arc to *maximize* the area to be occupied to found Carthage. Although the story of the founding of Carthage is fictitious, it probably inspired a new mathematical discipline, the *calculus of variations* and its extensions such as optimal control theory.

The calculus of variations is that branch of mathematics that deals with finding a function which is an extremum (maximum or minimum) of a functional. A functional is loosely defined as a function of a function. The theory of finding maxima and minima of functions is quite old and can be traced back to the isoperimetric problems considered by Greek mathematicians such as Zenodorus (495-435 B.C.) and by Poppus (c. 300 A.D.). But we will start with the works of Bernoulli. In 1699 Johannes Bernoulli (1667-1748) posed the brachistochrone problem: *the problem of finding the path of quickest descent between two points not in the same horizontal or vertical line.* This problem which was first posed by Galileo (1564-1642) in 1638, was solved by John, his brother Jacob (1654- 1705), by Gottfried Leibniz (1646-1716), and anonymously by Isaac Newton (1642-1727). Leonard Euler (1707-1783) joined John Bernoulli and made some remarkable contributions, which influenced Joseph-Louis Lagrange (1736-1813), who finally gave an el-

[1]The permission given by Springer-Verlag for H. H. Goldstine, *A History of the Calculus of Variations*, Springer-Verlag, New York, NY, 1980, is hereby acknowledged.

egant way of solving these types of problems by using the method of (*first*) variations. This led Euler to coin the phrase *calculus of variations.* Later this *necessary* condition for extrema of a functional was called the Euler - the Lagrange equation. Lagrange went on to treat variable end - point problems introducing the multiplier method, which later became one of the most powerful tools-Lagrange (or Euler-Lagrange) multiplier method-in optimization.

The *sufficient* conditions for finding the extrema of functionals in calculus of variations was given by Andrien Marie Legendre (1752-1833) in 1786 by considering additionally the *second* variation. Carl Gustav Jacob Jacobi (1804-1851) in 1836 came up with a more rigorous analysis of the sufficient conditions. This sufficient condition was later on termed as the Legendre-Jacobi condition. At about the same time Sir William Rowan Hamilton (1788-1856) did some remarkable work on mechanics, by showing that the motion of a particle in space, acted upon by various external forces, could be represented by a single function which satisfies *two* first-order partial differential equations. In 1838 Jacobi had some objections to this work and showed the need for only *one* partial differential equation. This equation, called Hamilton-Jacobi equation, later had profound influence on the calculus of variations and dynamic programming, optimal control, and as well as on mechanics.

The distinction between *strong* and *weak* extrema was addressed by Karl Weierstrass (1815-1897) who came up with the idea of the field of extremals and gave the Weierstrass condition, and sufficient conditions for weak and strong extrema. Rudolph Clebsch (1833-1872) and Adolph Mayer proceeded with establishing conditions for the more general class of problems. Clebsch formulated a problem in the calculus of variations by adjoining the constraint conditions in the form of differential equations and provided a condition based on second variation. In 1868 Mayer reconsidered Clebsch's work and gave some elegant results for the general problem in the calculus of variations. Later Mayer described in detail the problems: the problem of Lagrange in 1878, and the problem of Mayer in 1895.

In 1898, Adolf Kneser gave a new approach to the calculus of variations by using the result of Karl Gauss (1777-1855) on geodesics. For variable end-point problems, he established the transversality condition which includes orthogonality as a special case. He along with Oskar Bolza (1857-1942) gave sufficiency proofs for these problems. In 1900, David Hilbert (1862-1943) showed the second variation as a quadratic functional with eigenvalues and eigenfunctions. Between 1908 and 1910, Gilbert Bliss (1876-1951) [23] and Max Mason looked in depth at the results of Kneser. In 1913, Bolza formulated the problem of Bolza as a generalization of the problems of Lagrange and Mayer. Bliss showed that these three problems are equivalent. Other notable contributions to calculus of variations were made by E. J. McShane (1904-1989) [98], M. R. Hestenes [65], H. H. Goldstine and others. There have been a large number of books on the subject of calculus of variations: Bliss (1946) [23], Cicala (1957) [37], Akhiezer (1962) [1], Elsgolts (1962) [47], Gelfand and Fomin (1963) [55], Dreyfus (1966) [45], Forray (1968) [50], Balakrishnan (1969) [8], Young (1969) [146], Elsgolts (1970) [46], Bolza (1973) [26], Smith (1974) [126], Weinstock (1974) [143], Krasnov *et al.* (1975) [81], Leitmann (1981) [88], Ewing (1985) [48], Kamien and Schwartz (1991) [78], Gregory and Lin (1992) [61], Sagan (1992) [118], Pinch (1993) [108], Wan (1994) [141], Giaquinta and Hildebrandt (1995) [56, 57], Troutman (1996) [136], and Milyutin and Osmolovskii (1998) [103].

1.4.2 Optimal Control Theory

The linear quadratic control problem has its origins in the celebrated work of N. Wiener on mean-square filtering for weapon fire control during World War II (1940-45) [144, 145]. Wiener solved the problem of designing filters that minimize a mean-square-error criterion (performance measure) of the form

$$J = E\{e^2(t)\} \tag{1.4.1}$$

where, $e(t)$ is the error, and $E\{x\}$ represents the expected value of the random variable x. For a deterministic case, the above error criterion is generalized as an integral quadratic term as

$$J = \int_0^\infty \mathbf{e}'(t)\mathbf{Q}\mathbf{e}(t)dt \tag{1.4.2}$$

where, $\mathbf{Q}$ is some positive definite matrix. R. Bellman in 1957 [12] introduced the technique of *dynamic programming* to solve discrete-time optimal control problems. But, the most important contribution to optimal control systems was made in 1956 [25] by L. S. Pontryagin (formerly of the United Soviet Socialistic Republic (USSR)) and his associates, in development of his celebrated *maximum principle* described in detail in their book [109]. Also, see a very interesting article on the "discovery of the Maximum Principle" by R. V. Gamkrelidze [52], one of the authors of the original book [109]. At this time in the United States, R. E. Kalman in 1960 [70] provided *linear quadratic regulator (LQR)* and *linear quadratic Gaussian (LQG)* theory to design optimal feedback controls. He went on to present optimal filtering and estimation theory leading to his famous *discrete Kalman filter* [71] and the *continuous Kalman filter* with Bucy [76]. Kalman had a profound effect on optimal control theory and the Kalman filter is one of the most widely used technique in applications of control theory to real world problems in a variety of fields.

At this point we have to mention the *matrix Riccati equation* that appears in all the Kalman filtering techniques and many other fields. C. J. Riccati [114, 22] published his result in 1724 on the solution for some types of nonlinear differential equations, without ever knowing that the Riccati equation would become so famous after more than two centuries!

Thus, optimal control, having its roots in calculus of variations developed during 16th and 17th centuries was really born over 300 years ago [132]. For additional details about the historical perspectives on calculus of variations and optimal control, the reader is referred to some excellent publications [58, 99, 28, 21, 132].

In the so-called *linear quadratic control*, the term "linear" refers to the plant being *linear* and the term "quadratic" refers to the performance index that involves the *square* or *quadratic* of an error, and/or control. Originally, this problem was called the *mean-square* control problem and the term "linear quadratic" did not appear in the literature until the late 1950s.

Basically the *classical* control theory using *frequency* domain deals with single input and single output (SISO) systems, whereas *modern* control theory works with *time* domain for SISO and multi-input and multi-output (MIMO) systems. Although modern control and hence optimal control appeared to be very attractive, it lacked a very important feature of *robustness*. That is, controllers designed based on LQR theory failed to be robust to measurement noise, external disturbances and unmodeled dynamics. On the other hand, frequency domain techniques using the ideas of gain margin and phase margin offer robustness in a natural way. Thus, some researchers [115, 95], especially in the United Kingdom, continued to work on developing frequency domain approaches to MIMO systems.

One important and relevant field that has been developed around the 1980s is the $\mathcal{H}_\infty$-optimal control theory. In this framework, the work developed in the 1960s and 1970s is labeled as $\mathcal{H}_2$-optimal control theory. The seeds for $\mathcal{H}_\infty$-optimal control theory were laid by G. Zames [148], who formulated the optimal $\mathcal{H}_\infty$-sensitivity design problem for SISO systems and solved using optimal Nevanilina-Pick interpolation theory. An important publication in this field came from a group of four active researchers, Doyle, Glover, Khargonekar, and Francis[44], who won the 1991 W. R. G. Baker Award as the best IEEE Transactions paper. There are many other works in the field of $\mathcal{H}_\infty$ control ([51, 96, 43, 128, 7, 60, 131, 150, 39, 34]).

1.5 About This Book

This book, on the subject of optimal control systems, is based on the author's lecture notes used for teaching a graduate level course on this subject. In particular, this author was most influenced by Athans and Falb [6], Schultz and Melsa [121], Sage [119], Kirk [79], Sage and White [120], Anderson and Moore [3] and Lewis and Syrmos [91], and one finds the footprints of these works in the present book.

There were a good number of books on optimal control published during the era of the "glory of modern control," (Leitmann (1964) [87], Tou (1964) [135], Athans and Falb (1966) [6], Dreyfus (1966) [45], Lee and Markus (1967) [86], Petrov (1968) [106], Sage (1968) [119], Citron (1969) [38], Luenberger (1969) [93], Pierre (1969) [107], Pun (1969) [110], Young (1969) [146], Kirk (1970) [79], Boltyanskii [24], Kwakernaak and Sivan (1972) [84], Warga (1972) [142], Berkovitz (1974) [17], Bryson and Ho (1975) [30]), Sage and White (1977) [120], Leitmann (1981) [88]), Ryan (1982) [116]). There has been renewed interest with the second wave of books published during the last few years (Lewis (1986) [89], Stengal (1986) [127], Christensen *et al.* (1987) [36] Anderson and Moore (1990) [3], Hocking (1991) [66], Teo *et al.* (1991) [133], Gregory and Lin (1992) [61], Lewis (1992) [90], Pinch (1993) [108], Dorato *et al.* (1995) [42], Lewis and Syrmos (1995) [91]), Saberi *et al.* (1995) [117], Sima (1996) [124], Siouris [125], Troutman (1996) [136] Bardi and Dolcetta (1997) [9], Vincent and Grantham (1997) [139], Milyutin and Osmolovskii (1998) [103], Bryson (1999) [29], Burl [32], Kolosov (1999) [80], Pytlak (1999) [111], Vinter (2000) [140], Zelikin

(2000) [149], Betts (2001) [20], and Locatelli (2001) [92].

The optimal control theory continues to have a wide variety of applications starting from the traditional electrical power [36] to economics and management [16, 122, 78, 123].

1.6 Chapter Overview

This book is composed of seven chapters. Chapter 2 presents optimal control via calculus of variations. In this chapter, we start with some basic definitions and a simple variational problem of extremizing a functional. We then bring in the plant as a conditional optimization problem and discuss various types of problems based on the boundary conditions. We briefly mention both Lagrangian and Hamiltonian formalisms for optimization. Next, Chapter 3 addresses basically the linear quadratic regulator (LQR) system. Here we discuss the closed-loop optimal control system introducing matrix Riccati differential and algebraic equations. We look at the analytical solution to the Riccati equations and development of MATLAB© routine for the analytical solution. Tracking and other problems of linear quadratic optimal control are discussed in Chapter 4. We also discuss the gain and phase margins of the LQR system.

So far the optimal control of continuous-time systems is described. Next, the optimal control of discrete-time systems is presented in Chapter 5. Here, we start with the basic calculus of variations and then touch upon all the topics discussed above with respect to the continuous-time systems. The Pontryagin Principle and associated topics of dynamic programming and Hamilton-Jacobi-Bellman results are briefly covered in Chapter 6. The optimal control of systems with control and state constraints is described in Chapter 7. Here, we cover topics of control constraints leading to time-optimal, fuel-optimal and energy-optimal control systems and briefly discuss the state constraints problem.

Finally, the Appendices A and B provide summary of results on matrices, vectors, matrix algebra and state space, and Appendix C lists some of the MATLAB© files used in the book.

1.7 Problems

Problem 1.1 A D.C. motor speed control system is described by a second order state equation,

$$\dot{x}_1(t) = 25x_2(t)$$
$$\dot{x}_2(t) = -400x_1(t) - 200x_2(t) + 400u(t),$$

where, $x_1(t)$ = the speed of the motor, and $x_2(t)$ = the current in the armature circuit and the control input $u(t)$ = the voltage input to an amplifier supplying the motor. Formulate a performance index and optimal control problem to keep the speed constant at a particular value.

Problem 1.2 [83] In a liquid-level control system for a storage tank, the valves connecting a reservoir and the tank are controlled by gear train driven by a D. C. motor and an electronic amplifier. The dynamics is described by a third order system

$$\dot{x}_1(t) = -2x_1(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\dot{x}_3(t) = -10x_3(t) + 9000u(t)$$

where, $x_1(t)$ = is the height in the tank, $x_2(t)$ = is the angular position of the electric motor driving the valves controlling the liquid from reservoir to tank, $x_3(t)$ = the angular velocity of the motor, and $u(t)$ = is the input to electronic amplifier connected to the input of the motor. Formulate optimal control problem to keep the liquid level constant at a reference value and the system to act only if there is a change in the liquid level.

Problem 1.3 [35] In an inverted pendulum system, it is required to maintain the upright position of the pendulum on a cart. The linearized state equations are

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_3(t) + 0.2u(t)$$
$$\dot{x}_3(t) = x_4(t)$$
$$\dot{x}_4(t) = 10x_3(t) - 0.2u(t)$$

where, $x_1(t) =$ is horizontal linear displacement of the cart, $x_2(t) =$ is linear velocity of the cart, $x_3(t) =$ is angular position of the pendulum from vertical line, $x_4(t) =$ is angular velocity, and $u(t) =$ is the horizontal force applied to the cart. Formulate a performance index to keep the pendulum in the vertical position with as little energy as possible.

Problem 1.4 [101] A mechanical system consisting of two masses and two springs, one spring connecting the two masses and the other spring connecting one of the masses to a fixed point. An input is applied to the mass not connected to the fixed point. The displacements ($x_1(t)$ and $x_2(t)$) and the corresponding velocities ($x_3(t)$ and $x_4(t)$) of the two masses provide a fourth-order system described by

$$\begin{aligned}\dot{x}_1(t) &= x_3(t)\\ \dot{x}_2(t) &= x_4(t)\\ \dot{x}_3(t) &= -4x_1(t) + 2x_2(t)\\ \dot{x}_4(t) &= x_1(t) - x_2(t) + u(t)\end{aligned}$$

Formulate a performance index to minimize the errors in displacements and velocities and to minimize the control effort.

Problem 1.5 A simplified model of an automobile suspension system is described by

$$m\ddot{x}(t) + kx(t) = bu(t)$$

where, $x(t)$ is the position, $u(t)$ is the input to the suspension system (in the form of an upward force), m is the mass of the suspension system, and k is the spring constant. Formulate the optimal control problem for minimum control energy and passenger comfort. Assume suitable values for all the constants.

Problem 1.6 [112] Consider a continuous stirred tank chemical reactor described by

$$\begin{aligned}\dot{x}_1(t) &= -0.1x_1(t) - 0.12x_2(t)\\ \dot{x}_2(t) &= -0.3x_1(t) - 0.012x_2(t) - 0.07u(t)\end{aligned}$$

where, the normalized deviation state variables of the linearized model are $x_1(t)$ = reaction variable, $x_2(t)$ = temperature and the control variable $u(t)$ = effective cooling rate coefficient. Formulate a suitable performance measure to minimize the deviation errors and to minimize the control effort.

Chapter 2

Calculus of Variations and Optimal Control

Calculus of variations (CoV) or variational calculus deals with finding the *optimum* (maximum or minimum) value of a functional. Variational calculus that originated around 1696 became an independent mathematical discipline after the fundamental discoveries of L. Euler (1709-1783), whom we can claim with good reason as the founder of calculus of variations.

In this chapter, we start with some basic definitions and a simple variational problem of extremizing a functional. We then incorporate the plant as a conditional optimization problem and discuss various types of problems based on the boundary conditions. We briefly mention both the Lagrangian and Hamiltonian formalisms for optimization. It is suggested that the student reviews the material in Appendices A and B given at the end of the book. This chapter is motivated by [47, 79, 46, 143, 81, 48][1].

2.1 Basic Concepts

2.1.1 Function and Functional

We discuss some fundamental concepts associated with *functionals* along side with those of *functions.*

(a) Function: A variable x is a *function* of a variable quantity t, (writ-

[1]The permission given by Prentice Hall for D. E. Kirk, *Optimal Control Theory: An Introduction*, Prentice Hall, Englewood Cliffs, NJ, 1970, is hereby acknowledged.

ten as $x(t) = f(t)$), if to every value of t over a certain range of t there corresponds a value x; i.e., we have a correspondence: to a number t there corresponds a number x. Note that here t need not be always time but any independent variable.

Example 2.1

Consider

$$x(t) = 2t^2 + 1. \tag{2.1.1}$$

For $t = 1, x = 3, t = 2, x = 9$ and so on. Other functions are $x(t) = 2t$; $x(t_1, t_2) = t_1^2 + t_2^2$.

Next we consider the definition of a *functional* based on that of a function.

(b) Functional: A variable quantity J is a *functional* dependent on a function $f(x)$, written as $J = J(f(x))$, if to each function $f(x)$, there corresponds a value J, i.e., we have a correspondence: to the function $f(x)$ there corresponds a number J. Functional depends on several functions.

Example 2.2

Let $x(t) = 2t^2 + 1$. Then

$$J(x(t)) = \int_0^1 x(t)dt = \int_0^1 (2t^2 + 1)dt = \frac{2}{3} + 1 = \frac{5}{3} \tag{2.1.2}$$

is the area under the curve $x(t)$. If $v(t)$ is the velocity of a vehicle, then

$$J(v(t)) = \int_{t_0}^{t_f} v(t)dt \tag{2.1.3}$$

is the path traversed by the vehicle. Thus, here $x(t)$ and $v(t)$ are functions of t, and J is a functional of $x(t)$ or $v(t)$.

Loosely speaking, a functional can be thought of as a "function of a function."

2.1.2 Increment

We consider here *increment* of a function and a functional.

(a) Increment of a Function: In order to consider optimal values of a function, we need the definition of an increment [47, 46, 79].

DEFINITION 2.1 *The increment of the function f, denoted by Δf, is defined as*

$$\Delta f \triangleq f(t + \Delta t) - f(t). \tag{2.1.4}$$

It is easy to see from the definition that Δf depends on both the independent variable t and the increment of the independent variable Δt, and hence strictly speaking, we need to write the increment of a function as $\Delta f(t, \Delta t)$.

Example 2.3

If

$$f(t) = (t_1 + t_2)^2 \tag{2.1.5}$$

find the increment of the function $f(t)$.

Solution: The increment Δf becomes

$$\begin{aligned}
\Delta f &\triangleq f(t + \Delta t) - f(t) \\
&= (t_1 + \Delta t_1 + t_2 + \Delta t_2)^2 - (t_1 + t_2)^2 \\
&= (t_1 + \Delta t_1)^2 + (t_2 + \Delta t_2)^2 + 2(t_1 + \Delta t_1)(t_2 + \Delta t_2) - \\
&\quad (t_1^2 + t_2^2 + 2t_1 t_2) \\
&= 2(t_1 + t_2)\Delta t_1 + 2(t_1 + t_2)\Delta t_2 + (\Delta t_1)^2 + (\Delta t_2)^2 \\
&\quad + 2\Delta t_1 \Delta t_2.
\end{aligned} \tag{2.1.6}$$

(b) Increment of a Functional: Now we are ready to define the increment of a functional.

DEFINITION 2.2 *The increment of the functional J, denoted by ΔJ, is defined as*

$$\boxed{\Delta J \triangleq J(x(t) + \delta x(t)) - J(x(t)).} \tag{2.1.7}$$

Here $\delta x(t)$ is called the *variation* of the function $x(t)$. Since the increment of a functional is dependent upon the function $x(t)$ and its variation $\delta x(t)$, strictly speaking, we need to write the increment as $\Delta J(x(t), \delta x(t))$.

Example 2.4

Find the increment of the functional

$$J = \int_{t_0}^{t_f} \left[2x^2(t) + 1\right] dt. \tag{2.1.8}$$

Solution: The increment of J is given by

$$\begin{aligned}\Delta J &\triangleq J(x(t) + \delta x(t)) - J(x(t)),\\ &= \int_{t_0}^{t_f} \left[2(x(t) + \delta x(t))^2 + 1\right] dt - \int_{t_0}^{t_f} \left[2x^2(t) + 1\right] dt,\\ &= \int_{t_0}^{t_f} \left[4x(t)\delta x(t) + 2(\delta x(t))^2\right] dt. \end{aligned} \tag{2.1.9}$$

2.1.3 Differential and Variation

Here, we consider the *differential* of a function and the *variation* of a functional.

(a) Differential of a Function: Let us define at a point t^* the increment of the function f as

$$\Delta f \triangleq f(t^* + \Delta t) - f(t^*). \tag{2.1.10}$$

By expanding $f(t^* + \Delta t)$ in a Taylor series about t^*, we get

$$\Delta f = f(t^*) + \left(\frac{df}{dt}\right)_* \Delta t + \frac{1}{2!}\left(\frac{d^2 f}{dt^2}\right)_* (\Delta t)^2 + \cdots - f(t^*). \tag{2.1.11}$$

Neglecting the higher order terms in Δt,

$$\Delta f = \left(\frac{df}{dt}\right)_* \Delta t = \dot{f}(t^*)\Delta t = df. \tag{2.1.12}$$

Here, df is called the *differential* of f at the point t^*. $\dot{f}(t^*)$ is the *derivative* or slope of f at t^*. In other words, the differential df is the first order approximation to increment Δt. Figure 2.1 shows the relation between increment, differential and derivative.

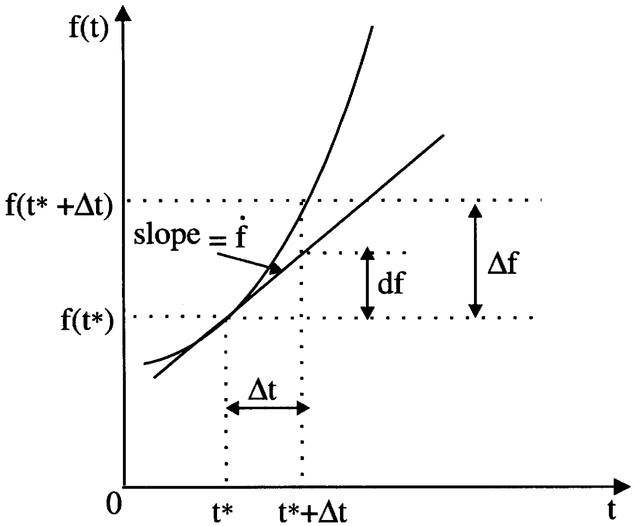

Figure 2.1 Increment Δf, Differential df, and Derivative $\dot{f}$ of a Function $f(t)$

Example 2.5

Let $f(t) = t^2 + 2t$. Find the increment and the derivative of the function $f(t)$.

Solution: By definition, the increment Δf is

$$\begin{aligned}
\Delta f &\triangleq f(t+\Delta t) - f(t), \\
&= (t+\Delta t)^2 + 2(t+\Delta t) - (t^2+2t), \\
&= 2t\Delta t + 2\Delta t + \cdots + \text{higher order terms}, \\
&= 2(t+1)\Delta t, \\
&= \dot{f}(t)\Delta t. \qquad (2.1.13)
\end{aligned}$$

Here, $\dot{f}(t) = 2(t+1)$.

(b) Variation of a Functional: Consider the increment of a functional

$$\Delta J \triangleq J(x(t) + \delta x(t)) - J(x(t)). \qquad (2.1.14)$$

Expanding $J(x(t) + \delta x(t))$ in a Taylor series, we get

$$\begin{aligned}\Delta J &= J(x(t)) + \frac{\partial J}{\partial x}\delta x(t) + \frac{1}{2!}\frac{\partial^2 J}{\partial x^2}(\delta x(t))^2 + \cdots - J(x(t)) \\ &= \frac{\partial J}{\partial x}\delta x(t) + \frac{1}{2!}\frac{\partial^2 J}{\partial x^2}(\delta x(t))^2 + \cdots \\ &= \delta J + \delta^2 J + \cdots, \end{aligned} \tag{2.1.15}$$

where,

$$\delta J = \frac{\partial J}{\partial x}\delta x(t) \quad \text{and} \quad \delta^2 J = \frac{1}{2!}\frac{\partial^2 J}{\partial x^2}(\delta x(t))^2 \tag{2.1.16}$$

are called the *first variation* (or simply the *variation*) and the *second variation* of the functional J, respectively. The variation δJ of a functional J is the *linear* (or first order approximate) part (in $\delta x(t)$) of the increment ΔJ. Figure 2.2 shows the relation between increment and the first variation of a functional.

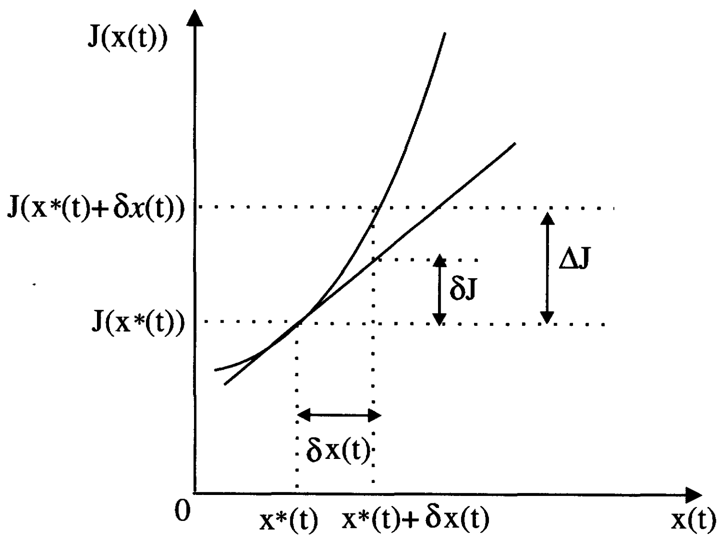

Figure 2.2 Increment ΔJ and the First Variation δJ of the Functional J

Example 2.6

Given the functional

$$J(x(t)) = \int_{t_0}^{t_f} [2x^2(t) + 3x(t) + 4]dt, \tag{2.1.17}$$

evaluate the variation of the functional.

Solution: First, we form the increment and then extract the variation as the first order approximation. Thus

$$\begin{aligned}\Delta J &\triangleq J(x(t) + \delta x(t)) - J(x(t)), \\ &= \int_{t_0}^{t_f} \Big[2(x(t) + \delta x(t))^2 + 3(x(t) + \delta x(t)) + 4) \\ &\quad -(2x^2(t) + 3x(t) + 4)\Big]\, dt, \\ &= \int_{t_0}^{t_f} \Big[4x(t)\delta x(t) + 2(\delta x(t))^2 + 3\delta x(t)\Big]\, dt. \end{aligned} \tag{2.1.18}$$

Considering only the first order terms, we get the (first) variation as

$$\delta J(x(t), \delta x(t)) = \int_{t_0}^{t_f} (4x(t) + 3)\delta x(t)dt. \tag{2.1.19}$$

2.2 Optimum of a Function and a Functional

We give some definitions for optimum or extremum (maximum or minimum) of a function and a functional [47, 46, 79]. The *variation* plays the same role in determining optimal value of a functional as the *differential* does in finding extremal or optimal value of a function.

DEFINITION 2.3 **Optimum of a Function:** *A function $f(t)$ is said to have a relative optimum at the point t^* if there is a positive parameter ϵ such that for all points t in a domain $\mathcal{D}$ that satisfy $|t - t^*| < \epsilon$, the increment of $f(t)$ has the same sign (positive or negative).*

In other words, if

$$\Delta f = f(t) - f(t^*) \geq 0, \tag{2.2.1}$$

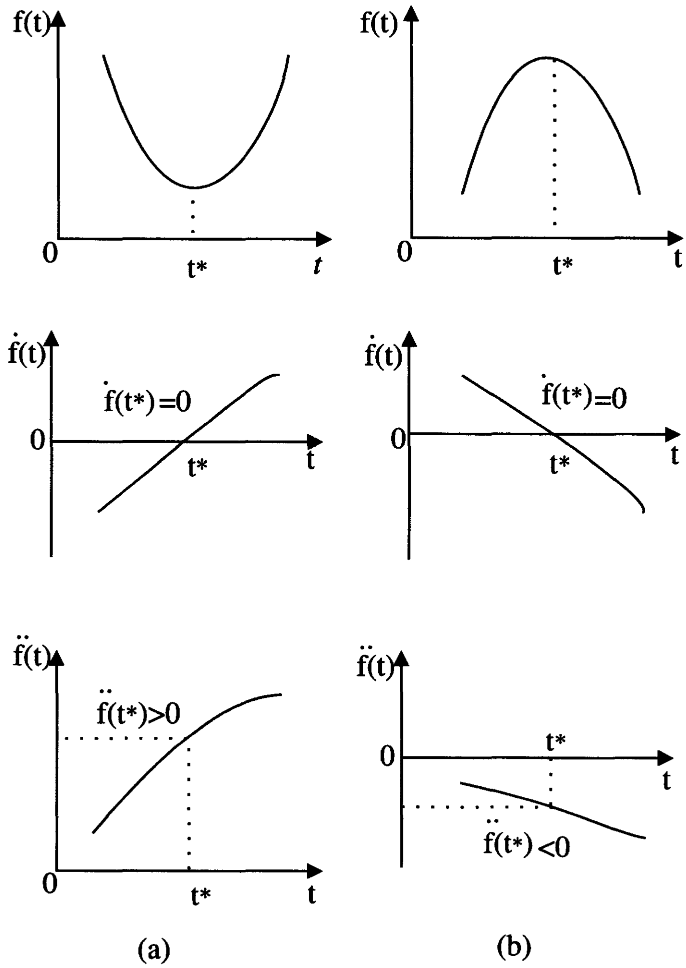

Figure 2.3 (a) Minimum and (b) Maximum of a Function $f(t)$

then, $f(t^*)$ is a relative local *minimum*. On the other hand, if

$$\Delta f = f(t) - f(t^*) \leq 0, \tag{2.2.2}$$

then, $f(t^*)$ is a relative local *maximum*. If the previous relations are valid for arbitrarily large ϵ, then, $f(t^*)$ is said to have a *global* absolute optimum. Figure 2.3 illustrates the (a) minimum and (b) maximum of a function.

It is well known that the *necessary* condition for optimum of a function is that the (first) differential vanishes, i.e., $df = 0$. The *sufficient* condition

1. for *minimum* is that the second differential is positive, i.e., $d^2 f > 0$, and
2. for *maximum* is that the second differential is negative, i.e., $d^2 f < 0$.

If $d^2 f = 0$, it corresponds to a *stationary* (or inflection) point.

DEFINITION 2.4 **Optimum of a Functional:** *A functional J is said to have a relative optimum at x^* if there is a positive ϵ such that for all functions x in a domain Ω which satisfy $|x - x^*| < \epsilon$, the increment of J has the same sign.*

In other words, if

$$\Delta J = J(x) - J(x^*) \geq 0, \tag{2.2.3}$$

then $J(x^*)$ is a relative *minimum*. On the other hand, if

$$\Delta J = J(x) - J(x^*) \leq 0, \tag{2.2.4}$$

then, $J(x^*)$ is a relative *maximum*. If the above relations are satisfied for arbitrarily large ϵ, then, $J(x^*)$ is a *global* absolute optimum.

Analogous to finding extremum or optimal values for *functions*, in variational problems concerning *functionals*, the result is that the variation must be zero on an optimal curve. Let us now state the result in the form of a theorem, known as *fundamental theorem of the calculus of variations*, the proof of which can be found in any book on calculus of variations [47, 46, 79].

THEOREM 2.1
For $x^*(t)$ to be a candidate for an optimum, the (first) variation of J must be zero on $x^*(t)$, i.e., $\delta J(x^*(t), \delta x(t)) = 0$ for all admissible values of $\delta x(t)$. This is a necessary *condition. As a* sufficient *condition for minimum, the second variation $\delta^2 J > 0$, and for maximum $\delta^2 J < 0$.*

2.3 The Basic Variational Problem

2.3.1 Fixed-End Time and Fixed-End State System

We address a fixed-end time and fixed-end state problem, where both the *initial* time and state and the *final* time and state are fixed or given *a priori.* Let $x(t)$ be a scalar function with continuous first derivatives and the vector case can be similarly dealt with. The problem is to find the *optimal* function $x^*(t)$ for which the functional

$$J(x(t)) = \int_{t_0}^{t_f} V(x(t), \dot{x}(t), t)dt \tag{2.3.1}$$

has a relative *optimum.* It is assumed that the integrand V has continuous first and second partial derivatives w.r.t. all its arguments; t_0 and t_f are fixed (or given a priori) and the end points are fixed, i.e.,

$$x(t = t_0) = x_0; \quad x(t = t_f) = x_f. \tag{2.3.2}$$

We already know from Theorem 2.1 that the necessary condition for an optimum is that the *variation of a functional vanishes.* Hence, in our attempt to find the optimum of $x(t)$, we first define the increment for J, obtain its variation and finally apply the fundamental theorem of the calculus of variations (Theorem 2.1).

Thus, the various steps involved in finding the optimal solution to the fixed-end time and fixed-end state system are first listed and then discussed in detail.

- **Step 1:** *Assumption of an Optimum*
- **Step 2:** *Variations and Increment*
- **Step 3:** *First Variation*
- **Step 4:** *Fundamental Theorem*
- **Step 5:** *Fundamental Lemma*
- **Step 6:** *Euler-Lagrange Equation*

- **Step 1:** *Assumption of an Optimum:* Let us assume that $x^*(t)$ is the optimum attained for the function $x(t)$. Take some admissible function $x_a(t) = x^*(t) + \delta x(t)$ close to $x^*(t)$, where $\delta x(t)$ is the variation of $x^*(t)$ as shown in Figure 2.4. The function $x_a(t)$ should also satisfy the boundary conditions (2.3.2) and hence it is necessary that

$$\delta x(t_0) = \delta x(t_f) = 0. \tag{2.3.3}$$

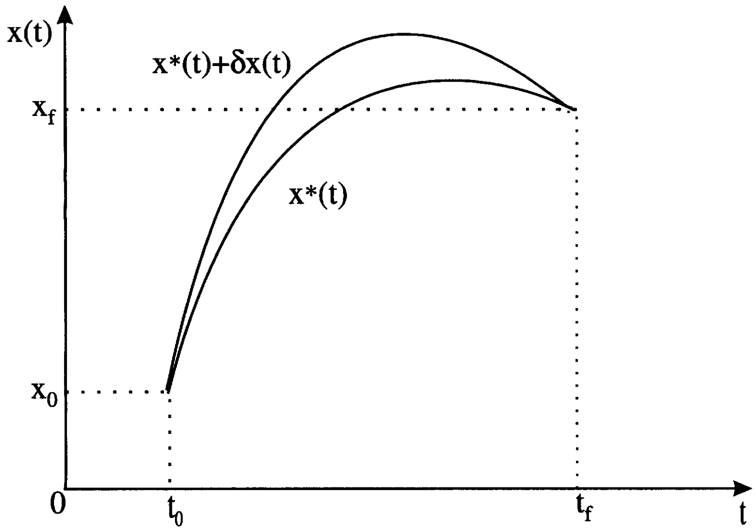

Figure 2.4 Fixed-End Time and Fixed-End State System

- **Step 2:** *Variations and Increment:* Let us first define the increment as

$$\begin{aligned}\Delta J(x^*(t), \delta x(t)) &\triangleq J(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t) \\ &\quad - J(x^*(t), \dot{x}^*(t), t) \\ &= \int_{t_0}^{t_f} V\left(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t\right) dt \\ &\quad - \int_{t_0}^{t_f} V(x^*(t), \dot{x}^*(t), t) dt. \qquad (2.3.4)\end{aligned}$$

which by combining the integrals can be written as

$$\begin{aligned}\Delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} &\left[V\left(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t\right)\right. \\ &\left. - V(x^*(t), \dot{x}^*(t), t)\right] dt. \qquad (2.3.5)\end{aligned}$$

where,

$$\dot{x}(t) = \frac{dx(t)}{dt} \quad \text{and} \quad \delta \dot{x}(t) = \frac{d}{dt}\left\{\delta x(t)\right\} \qquad (2.3.6)$$

Expanding V in the increment (2.3.5) in a Taylor series about the point $x^*(t)$ and $\dot{x}^*(t)$, the increment ΔJ becomes (note the cancelation of $V(x^*(t), \dot{x}^*(t), t))$

$$
\begin{aligned}
\Delta J &= \Delta J(x^*(t), \delta x(t)) \\
&= \int_{t_0}^{t_f} \left[\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial x} \delta x(t) + \frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}} \delta \dot{x}(t) \right. \\
&+ \frac{1}{2!} \left\{ \frac{\partial^2 V(...)}{\partial x^2} (\delta x(t))^2 + \frac{\partial^2 V(...)}{\partial \dot{x}^2} (\delta \dot{x}(t))^2 + \right. \\
&+ \left. \left. 2 \frac{\partial^2 V(...)}{\partial x \partial \dot{x}} \delta x(t) \delta \dot{x}(t) \right\} + \cdots \right] dt. \qquad (2.3.7)
\end{aligned}
$$

Here, the partial derivatives are w.r.t. $x(t)$ and $\dot{x}(t)$ at the optimal condition (*) and * is omitted for simplicity.

- **Step 3:** *First Variation:* Now, we obtain the variation by retaining the terms that are *linear* in $\delta x(t)$ and $\delta \dot{x}(t)$ as

$$
\begin{aligned}
\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} & \left[\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial x} \delta x(t) \right. \\
& \left. + \frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}} \delta \dot{x}(t) \right] dt. \qquad (2.3.8)
\end{aligned}
$$

To express the relation for the first variation (2.3.8) entirely in terms containing $\delta x(t)$ (since $\delta \dot{x}(t)$ is dependent on $\delta x(t)$), we integrate by parts the term involving $\delta \dot{x}(t)$ as (omitting the arguments in V for simplicity)

$$
\begin{aligned}
\int_{t_0}^{t_f} \left(\frac{\partial V}{\partial \dot{x}} \right)_* \delta \dot{x}(t) dt &= \int_{t_0}^{t_f} \left(\frac{\partial V}{\partial \dot{x}} \right)_* \frac{d}{dt} (\delta x(t)) dt \\
&= \int_{t_0}^{t_f} \left(\frac{\partial V}{\partial \dot{x}} \right)_* d(\delta x(t)), \\
&= \left[\left(\frac{\partial V}{\partial \dot{x}} \right)_* \delta x(t) \right]_{t_0}^{t_f} \\
&\quad - \int_{t_0}^{t_f} \delta x(t) \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{x}} \right)_* dt. \qquad (2.3.9)
\end{aligned}
$$

In the above, we used the well-known integration formula $\int u dv = uv - \int v du$ where $u = \partial V / \partial \dot{x}$ and $v = \delta x(t)$). Using (2.3.9), the relation (2.3.8) for first variation becomes

$$\begin{aligned}\delta J(x^*(t), \delta x(t)) &= \int_{t_0}^{t_f} \left(\frac{\partial V}{\partial x}\right)_* \delta x(t) dt + \left[\left(\frac{\partial V}{\partial \dot{x}}\right)_* \delta x(t)\right]_{t_0}^{t_f} \\ &\quad - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* \delta x(t) dt, \\ &= \int_{t_0}^{t_f} \left[\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_*\right] \delta x(t) dt \\ &\quad + \left[\left(\frac{\partial V}{\partial \dot{x}}\right)_* \delta x(t)\right]\Bigg|_{t_0}^{t_f}. \end{aligned} \tag{2.3.10}$$

Using the relation (2.3.3) for boundary variations in (2.3.10), we get

$$\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \left[\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_*\right] \delta x(t) dt. \tag{2.3.11}$$

- **Step 4:** *Fundamental Theorem:* We now apply the *fundamental theorem of the calculus of variations* (Theorem 2.1), i.e., the variation of J must vanish for an optimum. That is, for the optimum $x^*(t)$ to exist, $\delta J(x^*(t), \delta x(t)) = 0$. Thus the relation (2.3.11) becomes

$$\int_{t_0}^{t_f} \left[\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_*\right] \delta x(t) dt = 0. \tag{2.3.12}$$

Note that the function $\delta x(t)$ must be zero at t_0 and t_f, but for this, it is completely arbitrary.

- **Step 5:** *Fundamental Lemma:* To simplify the condition obtained in the equation (2.3.12), let us take advantage of the following lemma called the *fundamental lemma of the calculus of variations* [47, 46, 79].

LEMMA 2.1

If for every function $g(t)$ which is continuous,

$$\int_{t_0}^{t_f} g(t)\delta x(t) dt = 0 \tag{2.3.13}$$

where the function $\delta x(t)$ is continuous in the interval $[t_0, t_f]$, then the function $g(t)$ must be zero everywhere throughout the interval $[t_0, t_f]$. (see Figure 2.5.)

Proof: *We prove this by contradiction. Let us assume that $g(t)$ is nonzero (positive or negative) during a short interval $[t_a, t_b]$. Next, let us select $\delta x(t)$, which is arbitrary, to be positive (or negative) throughout the interval where $g(t)$ has a nonzero value. By this selection of $\delta x(t)$, the value of the integral in (2.3.13) will be nonzero. This contradicts our assumption that $g(t)$ is non-zero during the interval. Thus $g(t)$ must be identically zero everywhere during the entire interval $[t_0, t_f]$ in (2.3.13). Hence the lemma.*

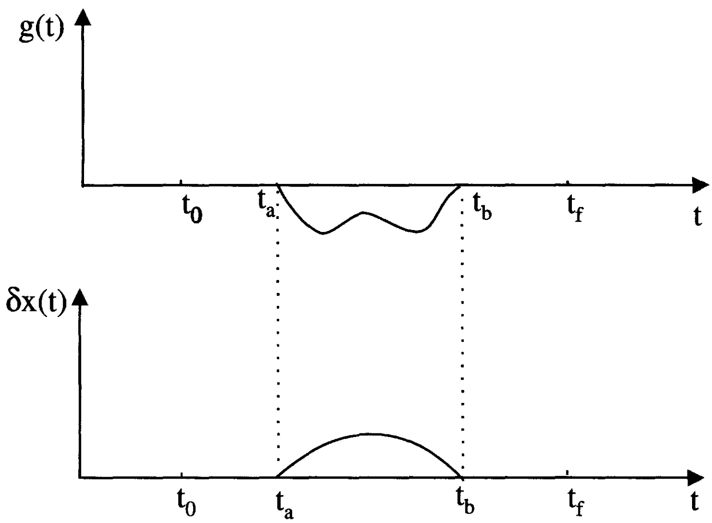

Figure 2.5 *A Nonzero $g(t)$ and an Arbitrary $\delta x(t)$*

- **Step 6:** *Euler-Lagrange Equation:* Applying the previous lemma to (2.3.12), a *necessary* condition for $x^*(t)$ to be an optimal of the functional J given by (2.3.1) is

$$\left(\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V(x^*(t), \dot{x}^*(t), t)}{\partial \dot{x}}\right)_* = 0 \quad (2.3.14)$$

or in simplified notation omitting the arguments in V,

$$\boxed{\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* = 0} \tag{2.3.15}$$

for all $t \in [t_0, t_f]$. This equation is called Euler equation, first published in 1741 [126].

A historical note is worthy of mention.

> *Euler obtained the equation (2.3.14) in 1741 using an elaborate and cumbersome procedure. Lagrange studied Euler's results and wrote a letter to Euler in 1755 in which he obtained the previous equation by a more elegant method of "variations" as described above. Euler recognized the simplicity and generality of the method of Lagrange and introduced the name* **calculus of variations**. *The all important fundamental equation (2.3.14) is now generally known as Euler-Lagrange (E.-L.) equation after these two great mathematicians of the 18th century. Lagrange worked further on optimization and came up with the well-known Lagrange multiplier rule or method.*

2.3.2 Discussion on Euler-Lagrange Equation

We provide some comments on the Euler-Lagrange equation [47, 46].

1. The Euler-Lagrange equation (2.3.14) can be written in many different forms. Thus (2.3.14) becomes

$$V_x - \frac{d}{dt}(V_{\dot{x}}) = 0 \tag{2.3.16}$$

where,

$$V_x = \frac{\partial V}{\partial x} = V_x(x^*(t), \dot{x}^*(t), t); \qquad V_{\dot{x}} = \frac{\partial V}{\partial \dot{x}} = V_{\dot{x}}(x^*(t), \dot{x}^*(t), t). \tag{2.3.17}$$

Since V is a function of three arguments $x^*(t)$, $\dot{x}^*(t)$, and t, and that $x^*(t)$ and $\dot{x}^*(t)$ are in turn functions of t, we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* &= \frac{d}{dt}\left(\frac{\partial V(x^*(t),\dot{x}^*(t),t)}{\partial \dot{x}}\right)_*, \\ &= \frac{d}{dt}\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}dx + \frac{\partial^2 V}{\partial \dot{x}\partial \dot{x}}d\dot{x} + \frac{\partial^2 V}{\partial t \partial \dot{x}}dt\right)_*, \\ &= \left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_*\left(\frac{dx}{dt}\right)_* + \left(\frac{\partial^2 V}{\partial \dot{x}\partial \dot{x}}\right)_*\left(\frac{d^2x}{dt^2}\right)_* + \left(\frac{\partial^2 V}{\partial t \partial \dot{x}}\right)_* \\ &= V_{x\dot{x}}\dot{x}^*(t) + V_{\dot{x}\dot{x}}\ddot{x}^*(t) + V_{t\dot{x}}. \end{aligned} \tag{2.3.18}$$

Combining (2.3.16) and (2.3.18), we get an alternate form for the EL equation as

$$\boxed{V_x - V_{t\dot{x}} - V_{x\dot{x}}\dot{x}^*(t) - V_{\dot{x}\dot{x}}\ddot{x}^*(t) = 0.} \tag{2.3.19}$$

2. The presence of $\frac{d}{dt}$ and/or $\dot{x}^*(t)$ in the EL equation (2.3.14) means that it is a *differential* equation.

3. In the EL equation (2.3.14), the term $\frac{\partial V(x^*(t),\dot{x}^*(t),t)}{\partial \dot{x}}$ is in general a function of $x^*(t)$, $\dot{x}^*(t)$, and t. Thus when this function is differentiated w.r.t. t, $\ddot{x}^*(t)$ may be present. This means that the differential equation (2.3.14) is in general of *second* order. This is also evident from the alternate form (2.3.19) for the EL equation.

4. There may also be terms involving products or powers of $\ddot{x}^*(t)$, $\dot{x}^*(t)$, and $x^*(t)$, in which case, the differential equation becomes *nonlinear.*

5. The explicit presence of t in the arguments indicates that the coefficients may be *time-varying.*

6. The conditions at initial point $t = t_0$ and final point $t = t_f$ leads to a *boundary value* problem.

7. Thus, the Euler-Lagrange equation (2.3.14) is, in general, a nonlinear, time-varying, two-point boundary value, second order, ordinary differential equation. Thus, we often have a nonlinear *two-point boundary value problem* (TPBVP). The solution of the nonlinear TPBVP is quite a formidable task and often done using numerical techniques. This is the price we pay for demanding optimal performance!

8. Compliance with the Euler-Lagrange equation is only a *necessary* condition for the optimum. Optimal may sometimes not yield either a maximum or a minimum; just as inflection points where the derivative vanishes in differential calculus. However, if the Euler-Lagrange equation is not satisfied for any function, this indicates that the optimum does not exist for that functional.

2.3.3 Different Cases for Euler-Lagrange Equation

We now discuss various cases of the EL equation.
Case 1: V is dependent of $\dot{x}(t)$, and t. That is, $V = V(\dot{x}(t), t)$. Then $V_x = 0$. The Euler-Lagrange equation (2.3.16) becomes

$$\frac{d}{dt}(V_{\dot{x}}) = 0. \tag{2.3.20}$$

This leads us to

$$V_{\dot{x}} = \frac{\partial V(\dot{x}^*(t), t)}{\partial \dot{x}} = C \tag{2.3.21}$$

where, C is a constant of integration.
Case 2: V is dependent of $\dot{x}(t)$ only. That is, $V = V(\dot{x}(t))$. Then $V_x = 0$. The Euler-Lagrange equation (2.3.16) becomes

$$\frac{d}{dt}(V_{\dot{x}}) = 0 \longrightarrow V_{\dot{x}} = C. \tag{2.3.22}$$

In general, the solution of either (2.3.21) or (2.3.22) becomes

$$\dot{x}^*(t) = C_1 \longrightarrow x^*(t) = C_1 t + C_2. \tag{2.3.23}$$

This is simply an equation of a straight line.
Case 3: V is dependent of $x(t)$ and $\dot{x}(t)$. That is, $V = V(x(t), \dot{x}(t))$. Then $V_{t\dot{x}} = 0$. Using the other form of the Euler-Lagrange equation (2.3.19), we get

$$V_x - V_{x\dot{x}}\dot{x}^*(t) - V_{\dot{x}\dot{x}}\ddot{x}^*(t) = 0. \tag{2.3.24}$$

Multiplying the previous equation by $\dot{x}^*(t)$, we have

$$\dot{x}^*(t)\left[V_x - V_{x\dot{x}}\dot{x}^*(t) - V_{\dot{x}\dot{x}}\ddot{x}^*(t)\right] = 0. \tag{2.3.25}$$

This can be rewritten as

$$\frac{d}{dt}(V - \dot{x}^*(t)V_{\dot{x}}) = 0 \longrightarrow V - \dot{x}^*(t)V_{\dot{x}} = C. \tag{2.3.26}$$

The previous equation can be solved using any of the techniques such as, separation of variables.

Case 4: V is dependent of $x(t)$, and t, i.e., $V = V(x(t), t)$. Then, $V_{\dot{x}} = 0$ and the Euler-Lagrange equation (2.3.16) becomes

$$\frac{\partial V(x^*(t), t)}{\partial x} = 0. \tag{2.3.27}$$

The solution of this equation does not contain any arbitrary constants and therefore generally speaking does not satisfy the boundary conditions $x(t_0)$ and $x(t_f)$. Hence, in general, no solution exists for this variational problem. Only in rare cases, when the function $x(t)$ satisfies the given boundary conditions $x(t_0)$ and $x(t_f)$, it becomes an optimal function.

Let us now illustrate the application of the EL equation with a very simple classic example of finding the shortest distance between two points. Often, we omit the $*$ (which indicates an optimal or extremal value) during the working of a problem and attach the same to the final solution.

Example 2.7

Find the minimum length between any two points.

Solution: It is well known that the solution to this problem is a straight line. However, we like to illustrate the application of Euler-Lagrange equation for this simple case. Consider the arc between two points A and B as shown in Figure 2.6. Let ds be the small arc length, and dx and dt are the small rectangular coordinate values. Note that t is the independent variable representing distance and not time. Then,

$$(ds)^2 = (dx)^2 + (dt)^2. \tag{2.3.28}$$

Rewriting

$$ds = \sqrt{1 + \dot{x}^2(t)}dt, \qquad \text{where } \dot{x}(t) = \frac{dx}{dt}. \tag{2.3.29}$$

Now the total arc length S between two points $x(t = t_0)$ and $x(t = t_f)$ is the performance index J to be minimized. Thus,

$$S = J = \int ds = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}^2(t)}dt = \int_{t_0}^{t_f} V(\dot{x}(t))dt \tag{2.3.30}$$

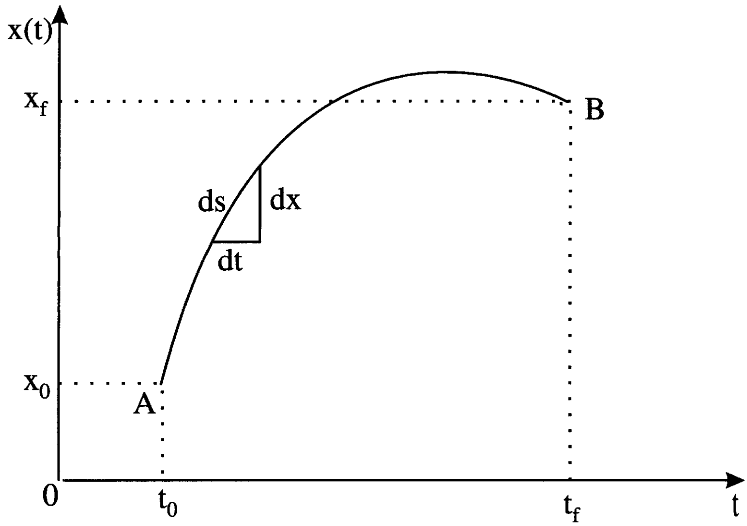

Figure 2.6 Arc Length

where, $V(\dot{x}(t)) = \sqrt{1 + \dot{x}^2(t)}$. Note that V is a function of $\dot{x}(t)$ only. Applying the Euler-Lagrange equation (2.3.22) to the performance index (2.3.30), we get

$$\frac{\dot{x}^*(t)}{\sqrt{1 + \dot{x}^{*2}(t)}} = C. \tag{2.3.31}$$

Solving this equation, we get the optimal solution as

$$x^*(t) = C_1 t + C_2. \tag{2.3.32}$$

This is evidently an equation for a straight line and the constants C_1 and C_2 are evaluated from the given boundary conditions. For example, if $x(0) = 1$ and $x(2) = 5$, $C_1 = 2$ and $C_2 = 1$ the straight line is $x^*(t) = 2t + 1$.

Although the previous example is a simple one,

1. it illustrates the formulation of a performance index from a given simple specification or a statement, and

2. the solution is well known *a priori* so that we can easily verify the application of the Euler-Lagrange equation.

In the previous example, we notice that the integrand V in the functional (2.3.30), is a function of $\dot{x}(t)$ only. Next, we take an example, where, V is a function of $x(t)$, $\dot{x}(t)$ and t.

Example 2.8

Find the optimum of

$$J = \int_0^2 \left[\dot{x}^2(t) - 2tx(t)\right] dt \tag{2.3.33}$$

that satisfy the boundary (initial and final) conditions

$$x(0) = 1 \quad \text{and} \quad x(2) = 5. \tag{2.3.34}$$

Solution: In the EL equation (2.3.19), we first identify that $V = \dot{x}^2(t) - 2tx(t)$. Then applying the EL equation (2.3.15) to the performance index (2.3.33) we get

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = 0 \longrightarrow -2t - \frac{d}{dt}\left(2\dot{x}(t)\right) = 0$$
$$\longrightarrow \ddot{x}(t) = t. \tag{2.3.35}$$

Solving the previous simple differential equation, we have

$$x^*(t) = \frac{t^3}{6} + C_1 t + C_2 \tag{2.3.36}$$

where, C_1 and C_2 are constants of integration. Using the given boundary conditions (2.3.19) in (2.3.36), we have

$$x(0) = 1 \longrightarrow C_2 = 1, \quad x(2) = 5 \longrightarrow C_1 = \frac{4}{3}. \tag{2.3.37}$$

With these values for the constants, we finally have the optimal function as

$$x^*(t) = \frac{t^3}{6} + \frac{4}{3}t + 1. \tag{2.3.38}$$

Another classical example in the calculus of variations is the *brachistochrone* (from *brachisto*, the shortest, and *chrones*, time) problem and this problem is dealt with in almost all books on calculus of variations [126].

Further, note that we have considered here only the so-called fixed-end point problem where both (initial and final) ends are fixed or given in advance. Other types of problems such as free-end point problems are not presented here but can be found in most of the books on the calculus of variations [79, 46, 81, 48]. However, these free-end point problems are better considered later in this chapter when we discuss the optimal control problem consisting of a performance index and a physical plant.

2.4 The Second Variation

In the study of extrema of functionals, we have so far considered only the *necessary* condition for a functional to have a relative or *weak extremum*, i.e., the condition that the first variation vanish leading to the classic *Euler-Lagrange equation.* To establish the nature of optimum (maximum or minimum), it is required to examine the *second variation.* In the relation (2.3.7) for the increment consider the terms corresponding to the second variation [120],

$$\delta^2 J = \int_{t_0}^{t_f} \frac{1}{2!}\left[\left(\frac{\partial^2 V}{\partial x^2}\right)_* (\delta x(t))^2 + \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* (\delta \dot{x}(t))^2 + 2\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_* \delta x(t)\delta \dot{x}(t)\right] dt. \tag{2.4.1}$$

Consider the last term in the previous equation and rewrite it in terms of $\delta x(t)$ only using integration by parts ($\int u dv = uv - \int v du$ where, $u = \frac{\partial^2 V}{\partial x \partial \dot{x}}\delta x(t)$ and $v = \delta x(t)$). Then using $\delta x(t_0) = \delta x(t_f) = 0$ for fixed-end conditions, we get

$$\delta^2 J = \frac{1}{2}\int_{t_0}^{t_f} \left[\left\{\left(\frac{\partial^2 V}{\partial x^2}\right)_* - \frac{d}{dt}\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_*\right\}(\delta x(t))^2 + \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* (\delta \dot{x}(t))^2\right] dt. \tag{2.4.2}$$

According to Theorem 2.1, the fundamental theorem of the calculus of variations, the sufficient condition for a *minimum* is $\delta^2 J > 0$. This, for arbitrary values of $\delta x(t)$ and $\delta \dot{x}(t)$, means that

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_* - \frac{d}{dt}\left(\frac{\partial^2 V}{\partial x \partial \dot{x}}\right)_* > 0, \tag{2.4.3}$$

$$\left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* > 0. \tag{2.4.4}$$

For *maximum*, the signs of the previous conditions are reversed. Alternatively, we can rewrite the second variation (2.4.1) in matrix form as

$$\begin{aligned}\delta^2 J &= \frac{1}{2}\int_{t_0}^{t_f} [\delta x(t)\ \delta \dot{x}(t)] \begin{bmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial \dot{x}} \\ \frac{\partial^2 V}{\partial x \partial \dot{x}} & \frac{\partial^2 V}{\partial \dot{x}^2}\end{bmatrix}_* \begin{bmatrix}\delta x(t) \\ \delta \dot{x}(t)\end{bmatrix} dt \\ &= \frac{1}{2}\int_{t_0}^{t_f} [\delta x(t)\ \delta \dot{x}(t)]\Pi \begin{bmatrix}\delta x(t) \\ \delta \dot{x}(t)\end{bmatrix} dt\end{aligned} \tag{2.4.5}$$

where,

$$\Pi = \begin{bmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial \dot{x}} \\ \\ \frac{\partial^2 V}{\partial x \partial \dot{x}} & \frac{\partial^2 V}{\partial \dot{x}^2} \end{bmatrix}_* . \tag{2.4.6}$$

If the matrix $\mathbf{\Pi}$ in the previous equation is positive (negative) definite, we establish a minimum (maximum). In many cases since $\delta x(t)$ is arbitrary, the coefficient of $(\delta \dot{x}(t))^2$, i.e., $\partial^2 V/\partial \dot{x}^2$ determines the sign of $\delta^2 J$. That is, the sign of second variation agrees with the sign of $\partial^2 V/\partial \dot{x}^2$. Thus, for *minimization* requirement

$$\boxed{\left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* > 0.} \tag{2.4.7}$$

For *maximization*, the sign of the previous equation reverses. In the literature, this condition is called *Legendre condition* [126].

> *In 1786, Legendre obtained this result of deciding whether a given optimum is maximum or minimum by examining the second variation. The second variation technique was further generalized by Jacobi in 1836 and hence this condition is usually called Legendre-Jacobi condition.*

Example 2.9

Verify that the straight line represents the minimum distance between two points.

Solution: This is an obvious solution, however, we illustrate the second variation. Earlier in Example 2.7, we have formulated a functional for the distance between two points as

$$J = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}^2(t)}dt = \int_{t_0}^{t_f} V(\dot{x}(t))dt \tag{2.4.8}$$

and found that the optimum is a straight line $x^*(t) = C_1 t + C_2$. To satisfy the sufficiency condition (2.4.7), we find

$$\left(\frac{\partial V}{\partial \dot{x}}\right)_* = \frac{\dot{x}^*(t)}{\sqrt{1 + \dot{x}^{*2}(t)}} \quad \text{and} \quad \left(\frac{\partial^2 V}{\partial \dot{x}^2}\right)_* = \frac{1}{\left[1 + \dot{x}^{*2}(t)\right]^{3/2}}. \tag{2.4.9}$$

Since $\dot{x}^*(t)$ is a constant (+ve or -ve), the previous equation satisfies the condition (2.4.7). Hence, the distance between two points as given by $x^*(t)$ (straight line) is minimum.

Next, we begin the second stage of optimal control. We consider optimization (or extremization) of a *functional* with a plant, which is considered as a constraint or a condition along with the functional. In other words, we address the extremization of a functional with some condition, which is in the form of a plant equation. The plant takes the form of state equation leading us to optimal control of dynamic systems. This section is motivated by [6, 79, 120, 108].

2.5 Extrema of Functions with Conditions

We begin with an example of finding the extrema of a function under a condition (or constraint). We solve this example with two methods, first by *direct* method and then by Lagrange *multiplier* method. Let us note that we consider this simple example only to illustrate some basic concepts associated with conditional extremization [120].

Example 2.10

A manufacturer wants to maximize the volume of the material stored in a circular tank subject to the condition that the material used for the tank is limited (constant). Thus, for a constant thickness of the material, the manufacturer wants to minimize the volume of the material used and hence part of the cost for the tank.

Solution: If a fixed metal thickness is assumed, this condition implies that the cross-sectional area of the tank material is constant. Let d and h be the diameter and the height of the circular tank. Then the volume contained by the tank is

$$V(d,h) = \pi d^2 h/4 \tag{2.5.1}$$

and the cross-sectional surface area (upper, lower and side) of the tank is

$$A(d,h) = 2\pi d^2/4 + \pi dh = A_0. \tag{2.5.2}$$

Our intent is to maximize $V(d,h)$ keeping $A(d,h) = A_0$, where A_0 is a given constant. We discuss two methods: first one is called the *Direct Method* using simple calculus and the second one is called *Lagrange Multiplier Method* using the Lagrange multiplier method.

1 Direct Method: In solving for the optimum value directly, we eliminate one of the variables, say h, from the volume relation (2.5.1) using the area relation (2.5.2). By doing so, the condition is *embedded* in the original function to be extremized. From (2.5.2),

$$h = \frac{A_0 - \pi d^2/2}{\pi d}. \tag{2.5.3}$$

Using the relation (2.5.3) for height in the relation (2.5.1) for volume

$$V(d) = A_0 d/4 - \pi d^3/8. \tag{2.5.4}$$

Now, to find the extrema of this simple calculus problem, we differentiate (2.5.4) w.r.t. d and set it to zero to get

$$\frac{A_0}{4} - \frac{3}{8}\pi d^2 = 0. \tag{2.5.5}$$

Solving, we get the optimal value of d as

$$d^* = \sqrt{\frac{2A_0}{3\pi}}. \tag{2.5.6}$$

By demanding that as per the Definition 2.3 for optimum of a function, the second derivative of V w.r.t. d in (2.5.4) be *negative* for *maximum*, we can easily see that the positive value of the square root function corresponds to the maximum value of the function. Substituting the optimal value of the diameter (2.5.6) in the original cross-sectional area given by (2.5.2), and solving for the optimum h^*, we get

$$h^* = \sqrt{\frac{2A_0}{3\pi}}. \tag{2.5.7}$$

Thus, we see from (2.5.6) and (2.5.7) that the volume stored by a tank is maximized if the height of the tank is made equal to its diameter.

2 Lagrange Multiplier Method: Now we solve the above problem by applying Lagrange multiplier method. We form a new function to be extremized by *adjoining* a given condition to the original function. The new adjoined function is extremized in the normal way by taking the partial derivatives w.r.t. all its variables, making them equal to zero, and solving for these variables which are extremals. Let the original volume relation (2.5.1) to be extremized be rewritten as

$$f(d,h) = \pi d^2 h/4 \tag{2.5.8}$$

and the condition (2.5.2) to be satisfied as

$$g(d,h) = 2\pi d^2/4 + \pi dh - A_0 = 0. \tag{2.5.9}$$

Then a new adjoint function $\mathcal{L}$ (called Lagrangian) is formed as

$$\begin{aligned}\mathcal{L}(d,h,\lambda) &= f(d,h) + \lambda g(d,h)\\ &= \pi d^2 h/4 + \lambda(2\pi d^2/4 + \pi dh - A_0)\end{aligned} \tag{2.5.10}$$

where, λ, a parameter yet to be determined, is called the *Lagrange multiplier*. Now, since the Lagrangian $\mathcal{L}$ is a function of three optimal variables d, h, and λ, we take the partial derivatives of $\mathcal{L}(d, h, \lambda)$ w.r.t. each of the variables d, h and λ and set them to zero. Thus,

$$\frac{\partial \mathcal{L}}{\partial d} = \pi dh/2 + \lambda(\pi d + \pi h) = 0 \tag{2.5.11}$$

$$\frac{\partial \mathcal{L}}{\partial h} = \pi d^2/4 + \lambda(\pi d) = 0 \tag{2.5.12}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2\pi d^2/4 + \pi dh - A_0 = 0. \tag{2.5.13}$$

Now, solving the previous three relations (2.5.11) to (2.5.13) for the three variables d^*, h^*, and λ^*, we get

$$d^* = \sqrt{\frac{2A_0}{3\pi}}; \quad h^* = \sqrt{\frac{2A_0}{3\pi}}; \quad \lambda^* = -\sqrt{\frac{A_0}{24\pi}}. \tag{2.5.14}$$

Once again, to maximize the volume of a cylindrical tank, we need to have the height (h^*) equal to the diameter (d^*) of the tank. Note that we need to take the negative value of the square root function for λ in (2.5.14) in order to satisfy the physical requirement that the diameter d obtained from (2.5.12) as

$$d = -4\lambda \tag{2.5.15}$$

is a positive value.

Now, we generalize the previous two methods.

2.5.1 Direct Method

Now we generalize the preceding method of elimination using differential calculus. Consider the extrema of a function $f(x_1, x_2)$ with two *interdependent* variables x_1 and x_2, subject to the condition

$$g(x_1, x_2) = 0. \tag{2.5.16}$$

As a necessary condition for extrema, we have

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0. \tag{2.5.17}$$

However, since dx_1 and dx_2 are not *arbitrary*, but related by the condition

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0, \tag{2.5.18}$$

it is not possible to conclude as in the case of extremization of functions without conditions that

$$\frac{\partial f}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = 0 \tag{2.5.19}$$

in the necessary condition (2.5.17). This is easily seen, since if the set of extrema conditions (2.5.19) is solved for optimal values x_1^* and x_2^*, there is no guarantee that these optimal values, would, in general satisfy the given condition (2.5.16).

In order to find optimal values that satisfy both the condition (2.5.16) and that of the extrema conditions (2.5.17), we arbitrarily choose one of the variables, say x_1, as the *independent* variable. Then x_2 becomes a *dependent* variable as per the condition (2.5.16). Now, assuming that $\partial g/\partial x_2 \neq 0$, (2.5.18) becomes

$$dx_2 = -\left\{\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\right\} dx_1 \tag{2.5.20}$$

and using (2.5.20) in the necessary condition (2.5.17), we have

$$df = \left[\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\left\{\frac{\partial g/\partial x_1}{\partial g/\partial x_2}\right\}\right] dx_1 = 0. \tag{2.5.21}$$

As we have chosen dx_1 to be the *independent*, we now can consider it to be *arbitrary*, and conclude that in order to satisfy (2.5.21), we have the coefficient of dx_1 to be zero. That is

$$\left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial g}{\partial x_2}\right) - \left(\frac{\partial f}{\partial x_2}\right)\left(\frac{\partial g}{\partial x_1}\right) = 0. \tag{2.5.22}$$

Now, the relation (2.5.22) and the condition (2.5.16) are solved simultaneously for the optimal solutions x_1^* and x_2^*. Equation (2.5.22) can be rewritten as

$$\begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{vmatrix} = 0. \tag{2.5.23}$$

This is also, as we know, the Jacobian of f and g w.r.t. x_1 and x_2. This method of elimination of the dependent variables is quite tedious for higher order problems.

2.5.2 Lagrange Multiplier Method

We now generalize the second method of solving the same problem of extrema of functions with conditions. Consider again the extrema of the function $f(x_1, x_2)$ subject to the condition

$$g(x_1, x_2) = 0. \tag{2.5.24}$$

In this method, we form an *augmented* Lagrangian function

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2) \tag{2.5.25}$$

where, λ, a parameter (multiplier) yet to be determined, is the Lagrange multiplier. Let us note that using the given condition (2.5.24) in the Lagrangian (2.5.25), we have

$$\mathcal{L}(x_1, x_2) = f(x_1, x_2) \tag{2.5.26}$$

and therefore a *necessary* condition for extrema is that

$$df = d\mathcal{L} = 0. \tag{2.5.27}$$

Accepting the idea that the Lagrangian (2.5.25) is a better representation of the entire problem than the equation (2.5.26) in finding the extrema, we have from the Lagrangian relation (2.5.25)

$$d\mathcal{L} = df + \lambda dg = 0. \tag{2.5.28}$$

Using (2.5.17) and (2.5.18) in (2.5.28), and rearranging

$$\left[\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\right] dx_1 + \left[\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2}\right] dx_2 = 0. \tag{2.5.29}$$

Now dx_1 and dx_2 are *both* not independent and hence cannot immediately conclude that each of the coefficients of dx_1 and dx_2 in (2.5.29) must be zero. Let us choose dx_1 to be *independent* differential and then dx_2 becomes a *dependent* differential as per (2.5.18). Further, let us choose the multiplier λ, which has been introduced by us and is at our disposal, to make one of the coefficients of dx_1 or dx_2 in (2.5.29) zero. For example, let λ take on the value λ^* that makes the coefficient of the *dependent* differential dx_2 equal zero, that is

$$\frac{\partial f}{\partial x_2} + \lambda^* \frac{\partial g}{\partial x_2} = 0. \tag{2.5.30}$$

With (2.5.30), the equation (2.5.29) reduces to

$$\left[\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\right] dx_1 = 0. \tag{2.5.31}$$

Since, dx_1 is the *independent* differential, it can be varied *arbitrarily.* Hence, for (2.5.31) to be satisfied for all dx_1, the coefficient of dx_1 must be zero. That is

$$\frac{\partial f}{\partial x_1} + \lambda^* \frac{\partial g}{\partial x_1} = 0. \tag{2.5.32}$$

Now from (2.5.25), note that

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \tag{2.5.33}$$

yields the constraint relation (2.5.16). Combining the results from (2.5.32), (2.5.30), and (2.5.33), we have

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda^* \frac{\partial g}{\partial x_1} = 0 \tag{2.5.34}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda^* \frac{\partial g}{\partial x_2} = 0 \tag{2.5.35}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0. \tag{2.5.36}$$

The preceding three equations are to be solved simultaneously to obtain x_1^*, x_2^*, and λ^*. By eliminating λ^* between (2.5.34) and (2.5.35)

$$\left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial g}{\partial x_2}\right) - \left(\frac{\partial f}{\partial x_2}\right)\left(\frac{\partial g}{\partial x_1}\right) = 0 \tag{2.5.37}$$

which is the same condition as (2.5.22) obtained by the direct method, thus indicating that we have the same result by Lagrange multiplier method.

Let us note that the necessary conditions (2.5.34) and (2.5.35) are just the same equations which would have been obtained from considering the differentials dx_1 and dx_2 as though they were *independent* in (2.5.29). Introduction of the multiplier λ has allowed us to treat all the variables in the augmented function $\mathcal{L}(x_1, x_2, \lambda)$ as though each variable is *independent.* Thus, the multiplier λ has acted like a *catalyst,* appearing in the intermediate stage only.

Summarizing, the extrema of a function $f(x_1, x_2)$ subject to the condition (or constraint) $g(x_1, x_2) = 0$ is equivalent to extrema of a single augmented function $\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$ as though x_1, x_2 and λ are *independent*. We now generalize this result.

THEOREM 2.2

Consider the extrema of a continuous, real-valued function $f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$ *subject to the conditions*

$$\begin{aligned} g_1(\mathbf{x}) &= g_1(x_1, x_2, \cdots, x_n) = 0 \\ g_2(\mathbf{x}) &= g_2(x_1, x_2, \cdots, x_n) = 0 \\ &\cdots \\ g_m(\mathbf{x}) &= g_m(x_1, x_2, \cdots, x_n) = 0 \end{aligned} \tag{2.5.38}$$

where, f *and* $\mathbf{g}$ *have continuous partial derivatives, and* $m < n$. *Let* $\lambda_1, \lambda_2, \cdots, \lambda_m$ *be the Lagrange multipliers corresponding to* m *conditions, such that the augmented Lagrangian function is formed as*

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}'\mathbf{g}(\mathbf{x}), \tag{2.5.39}$$

where, $\boldsymbol{\lambda}'$ *is the* transpose *of* $\boldsymbol{\lambda}$. *Then, the optimal values* $\mathbf{x}^*$ *and* $\boldsymbol{\lambda}^*$ *are the solutions of the following* $n + m$ *equations*

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}} + \boldsymbol{\lambda}'\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = 0 \tag{2.5.40}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} = \mathbf{g}(\mathbf{x}) = 0. \tag{2.5.41}$$

Features of Lagrange Multiplier: The Lagrange multiplier method is a powerful one in finding the extrema of functions subject to conditions. It has the following attractive features:

1. The importance of the Lagrange multiplier technique lies on the fact that the problem of determining the extrema of the function $f(\mathbf{x})$ subject to the conditions $\mathbf{g}(\mathbf{x}) = 0$ is *embedded* within the simple problem of determining the extrema of the simple *augmented* function $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}'\mathbf{g}(\mathbf{x})$.

2. Introduction of Lagrange multiplier allows us to treat all the variables $\mathbf{x}$ and $\boldsymbol{\lambda}$ in the augmented function $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$ as though each were *independent*.

3. The multiplier λ acts like a *catalyst* in the sense that it is introduced to perform a certain duty as given by item 2.

4. The increased dimensionality $(n+m)$ which is characteristic of the Lagrange multiplier method, is generally more than compensated by the relative *simplicity* and *systematic* procedure of the technique.

The multiplier method was given by Lagrange in 1788.

2.6 Extrema of Functionals with Conditions

In this section, we extend our ideas to functionals based on those developed in the last section for functions. First, we consider a functional with two variables, use the results of the previous section on the CoV, derive the necessary conditions and then extend the same for a general nth order vector case. Consider the extremization of the performance index in the form of a functional

$$J(x_1(t), x_2(t), t) = J = \int_{t_0}^{t_f} V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t)dt \quad (2.6.1)$$

subject to the condition (plant or system equation)

$$g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)) = 0 \quad (2.6.2)$$

with fixed-end-point conditions

$$\begin{aligned} x_1(t_0) &= x_{10}; \quad x_2(t_0) = x_{20} \\ x_1(t_f) &= x_{1f}; \quad x_2(t_f) = x_{2f}. \end{aligned} \quad (2.6.3)$$

Now we address this problem under the following steps.

- **Step 1:** *Lagrangian*
- **Step 2:** *Variations and Increment*
- **Step 3:** *First Variation*
- **Step 4:** *Fundamental Theorem*
- **Step 5:** *Fundamental Lemma*
- **Step 6:** *Euler-Lagrange Equation*

- **Step 1:** *Lagrangian:* We form an *augmented* functional

$$J_a = \int_{t_0}^{t_f} \mathcal{L}(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t), t)dt \qquad (2.6.4)$$

where, $\lambda(t)$ is the Lagrange multiplier, and the Lagrangian $\mathcal{L}$ is defined as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), \lambda(t), t) \\ &= V(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), t) \\ &\quad + \lambda(t) g(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)) \end{aligned} \qquad (2.6.5)$$

Note from the performance index (2.6.1) and the augmented performance index (2.6.4) that $J_a = J$ if the condition (2.6.2) is satisfied for any $\lambda(t)$.

- **Step 2:** *Variations and Increment:* Next, assume optimal values and then consider the *variations* and *increment* as

$$\begin{aligned} x_i(t) &= x_i^*(t) + \delta x_i(t), \qquad \dot{x}_i(t) = \dot{x}_i^*(t) + \delta \dot{x}_i(t), \quad i = 1, 2 \\ \Delta J_a &= J_a(x_i^*(t) + \delta x_i(t), \dot{x}_i^*(t) + \delta \dot{x}_i(t), t) - J_a(x_i^*(t), \dot{x}_i^*(t), t), \end{aligned} \qquad (2.6.6)$$

for $i = 1, 2$.

- **Step 3:** *First Variation:* Then using the Taylor series expansion and retaining linear terms only, the *first* variation of the functional J_a becomes

$$\begin{aligned} \delta J_a = \int_{t_0}^{t_f} \bigg[& \left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* \delta x_1(t) + \left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* \delta x_2(t) \\ & + \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta \dot{x}_1(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* \delta \dot{x}_2(t) \bigg] dt. \end{aligned} \qquad (2.6.7)$$

As before in the section on CoV, we rewrite the terms containing $\delta \dot{x}_1(t)$ and $\delta \dot{x}_2(t)$ in terms of those containing $\delta x_1(t)$ and $\delta x_2(t)$

only (using integration by parts, $\int u dv = uv - \int v du$). Thus

$$\begin{aligned}\int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta \dot{x}_1(t) dt &= \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \frac{d}{dt}(\delta x_1(t)) dt \\ &= \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* d(\delta x_1(t)) \\ &= \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta x_1(t)\right]\Bigg|_{t_0}^{t_f} \\ &\quad - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta x_1(t) dt.\end{aligned} \tag{2.6.8}$$

Using the above, we have the first variation (2.6.7) as

$$\begin{aligned}\delta J_a = &\int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* \delta x_1(t) + \left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* \delta x_2(t)\right] dt \\ &+ \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta x_1(t)\right]\Bigg|_{t_0}^{t_f} + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* \delta x_2(t)\right]\Bigg|_{t_0}^{t_f} \\ &- \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* \delta x_1(t) dt - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* \delta x_2(t) dt.\end{aligned} \tag{2.6.9}$$

Since this is a fixed-final time and fixed-final state problem as given by (2.6.3), no variations are allowed at the final point. This means

$$\delta x_1(t_0) = \delta x_2(t_0) = \delta x_1(t_f) = \delta x_2(t_f) = 0. \tag{2.6.10}$$

Using the boundary variations (2.6.10) in the augmented first variation (2.6.9), we have

$$\begin{aligned}\delta J_a = &\int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_*\right] \delta x_1(t) dt \\ &+ \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_*\right] \delta x_2(t) dt.\end{aligned} \tag{2.6.11}$$

- **Step 4:** *Fundamental Theorem:* Now, we proceed as follows.

  1. We invoke the fundamental theorem of the calculus of variations (Theorem 2.1) and make the first variation (2.6.11) equal to zero.

2. Remembering that *both* $\delta x_1(t)$ and $\delta x_2(t)$ are not independent, because $x_1(t)$ and $x_2(t)$ are related by the condition (2.6.2), we choose $\delta x_2(t)$ as the *independent* variation and $\delta x_1(t)$ as the *dependent* variation.
3. Let us choose the multiplier $\lambda^*(t)$ which is arbitrarily introduced and is at our disposal, in such a way that the coefficient of the *dependent* variation $\delta x_1(t)$ in (2.6.11) vanish. That is

$$\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* = 0. \tag{2.6.12}$$

With these choices, the first variation (2.6.11) becomes

$$\int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_*\right] \delta x_2(t) dt = 0. \tag{2.6.13}$$

- **Step 5:** *Fundamental Lemma:* Using the fundamental lemma of CoV (Lemma 2.1) and noting that since $\delta x_2(t)$ has been chosen to be *independent* variation and hence *arbitrary*, the only way (2.6.13) can be satisfied, in general, is that the coefficient of $\delta x_1(t)$ also vanish. That is

$$\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* = 0. \tag{2.6.14}$$

Also, from the Lagrangian(2.6.5) note that

$$\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* = 0 \tag{2.6.15}$$

yields the constraint relation (2.6.2).

- **Step 6:** *Euler-Lagrange Equation:* Combining the various relations (2.6.12), (2.6.14), and (2.6.15), the *necessary* conditions for extremization of the functional (2.6.1) subject to the condition (2.6.2) (according to Euler-Lagrange equation) are

$$\left(\frac{\partial \mathcal{L}}{\partial x_1}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right)_* = 0 \tag{2.6.16}$$

$$\left(\frac{\partial \mathcal{L}}{\partial x_2}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right)_* = 0 \tag{2.6.17}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\lambda}}\right)_* = 0. \tag{2.6.18}$$

Let us note that these conditions are just the ones that would have been obtained from the Lagrangian (2.6.5), as if both $\delta x_1(t)$ and $\delta x_2(t)$ had been *independent.* Also, in (2.6.18), the Lagrangian $\mathcal{L}$ is independent of $\dot{\lambda}(t)$ and hence the condition (2.6.18) is really the given plant equation (2.6.2).

Thus, the introduction of the Lagrange multiplier $\lambda(t)$ has enabled us to treat the variables $x_1(t)$ and $x_2(t)$ as though they were *independent*, in spite of the fact that they are related by the condition (2.6.2). The solution of the two, second-order differential equations (2.6.16) and (2.6.17) and the condition relation (2.6.2) or (2.6.18) along with the boundary conditions (2.6.3) give the optimal solutions $x_1^*(t)$, $x_2^*(t)$, and $\lambda^*(t)$.

Now, we generalize the preceding procedure for an nth order system. Consider the extremization of a functional

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt \tag{2.6.19}$$

where, $\mathbf{x}(t)$ is an nth order state vector, subject to the plant equation (condition)

$$g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0; \qquad i = 1, 2, \ldots, m \tag{2.6.20}$$

and boundary conditions, $\mathbf{x}(0)$ and $\mathbf{x}(t_f)$. We form an augmented functional

$$J_a = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t)dt \tag{2.6.21}$$

where, the Lagrangian $\mathcal{L}$ is given by

$$\boxed{\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \boldsymbol{\lambda}'(t) g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)} \tag{2.6.22}$$

and the Lagrange multiplier $\boldsymbol{\lambda}(t) = [\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t)]'$. We now apply the Euler-Lagrange equation on J_a to yield

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0,} \tag{2.6.23}$$

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}}\right)_* = 0 \longrightarrow g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0.} \tag{2.6.24}$$

Note that from (2.6.22), the Lagrangian $\mathcal{L}$ is independent of $\dot{\lambda}(t)$ and hence the Euler-Lagrange equation (2.6.24) is nothing but the given relation regarding the plant or the system (2.6.20). Thus, we solve the Euler-Lagrange equation (2.6.23) along with the given boundary conditions. Let us now illustrate the preceding method by a simple example.

Example 2.11

Minimize the performance index

$$J = \int_0^1 \left[x^2(t) + u^2(t)\right] dt \tag{2.6.25}$$

with boundary conditions

$$x(0) = 1; \quad x(1) = 0 \tag{2.6.26}$$

subject to the condition (plant equation)

$$\dot{x}(t) = -x(t) + u(t). \tag{2.6.27}$$

Solution: Let us solve this problem by the two methods, i.e., the direct method and the Lagrange multiplier method.

1 Direct Method: Here, we eliminate $u(t)$ between the performance index (2.6.25) and the plant (2.6.27) to get the functional as

$$\begin{aligned} J &= \int_0^1 [x^2(t) + (\dot{x}(t) + x(t))^2]dt \\ &= \int_0^1 [2x^2(t) + \dot{x}^2(t) + 2x(t)\dot{x}(t)]dt. \end{aligned} \tag{2.6.28}$$

Now, we notice that the functional (2.6.28) absorbed the condition (2.6.27) within itself, and we need to consider it as a straight forward extremization of a functional as given earlier. Thus, applying the Euler-Lagrange equation

$$\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* = 0 \tag{2.6.29}$$

to the functional (2.6.28), where,

$$V = 2x^2(t) + \dot{x}^2(t) + 2x(t)\dot{x}(t), \tag{2.6.30}$$

we get

$$4x^*(t) + 2\dot{x}^*(t) - \frac{d}{dt}(2\dot{x}^*(t) + 2x^*(t)) = 0. \tag{2.6.31}$$

Simplifying the above

$$\ddot{x}^*(t) - 2x^*(t) = 0 \tag{2.6.32}$$

the solution (see later for use of MATLAB©) of which gives the optimal as

$$x^*(t) = C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t} \tag{2.6.33}$$

where, the constants C_1 and C_2, evaluated using the given boundary conditions (2.6.26), are found to be

$$C_1 = 1/(1 - e^{-2\sqrt{2}}); \quad C_2 = 1/(1 - e^{2\sqrt{2}}). \tag{2.6.34}$$

Finally, knowing the optimal $x^*(t)$, the optimal control $u^*(t)$ is found from the plant (2.6.27) to be

$$\begin{aligned} u^*(t) &= \dot{x}^*(t) + x^*(t) \\ &= C_1(1 - \sqrt{2})e^{-\sqrt{2}t} + C_2(1 + \sqrt{2})e^{\sqrt{2}t}. \end{aligned} \tag{2.6.35}$$

Although the method appears to be simple, let us note that it is not always possible to eliminate $u(t)$ from (2.6.25) and (2.6.27) especially for higher-order systems.

2 Lagrange Multiplier Method: Here, we use the ideas developed in the previous section on the extremization of functions with conditions. Consider the optimization of the functional (2.6.25) with the boundary conditions (2.6.26) under the condition describing the plant (2.6.27). First we rewrite the condition (2.6.27) as

$$g(x(t), \dot{x}(t), u(t)) = \dot{x}(t) + x(t) - u(t) = 0. \tag{2.6.36}$$

Now, we form an augmented functional as

$$\begin{aligned} J &= \int_0^1 \left[x^2(t) + u^2(t) + \lambda(t)\left\{\dot{x}(t) + x(t) - u(t)\right\}\right] dt \\ &= \int_0^1 \mathcal{L}(x(t), \dot{x}(t), u(t), \lambda(t))dt \end{aligned} \tag{2.6.37}$$

where, $\lambda(t)$ is the Lagrange multiplier, and

$$\begin{aligned} \mathcal{L}(x(t), \dot{x}(t), u(t), \lambda(t)) = x^2(t) + u^2(t) \\ + \lambda(t)\left\{\dot{x}(t) + x(t) - u(t)\right\} \end{aligned} \tag{2.6.38}$$

is the Lagrangian. Now, we apply the Euler-Lagrange equation to the previous Lagrangian to get

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_* = 0 \longrightarrow 2x^*(t) + \lambda^*(t) - \dot{\lambda}^*(t) = 0 \quad (2.6.39)$$

$$\left(\frac{\partial \mathcal{L}}{\partial u}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right)_* = 0 \longrightarrow 2u^*(t) - \lambda^*(t) = 0 \quad (2.6.40)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\lambda}}\right)_* = 0 \longrightarrow \dot{x}^*(t) + x^*(t) - u^*(t) = 0 \quad (2.6.41)$$

and solve for optimal $x^*(t)$, $u^*(t)$, and $\lambda^*(t)$. We get first from (2.6.40) and (2.6.41)

$$\lambda^*(t) = 2u^*(t) = 2(\dot{x}^*(t) + x^*(t)). \quad (2.6.42)$$

Using the equation (2.6.42) in (2.6.39)

$$2x^*(t) + 2(\dot{x}^*(t) + x^*(t)) - 2(\ddot{x}^*(t) + \dot{x}^*(t)) = 0. \quad (2.6.43)$$

Solving the previous equation, we get

$$\ddot{x}^*(t) - 2x^*(t) = 0 \longrightarrow x^*(t) = C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t}. \quad (2.6.44)$$

Once we know $x^*(t)$, we get $\lambda^*(t)$ and hence $u^*(t)$ from (2.6.42) as

$$\begin{aligned} u^*(t) &= \dot{x}^*(t) + x^*(t) \\ &= C_1(1 - \sqrt{2})e^{-\sqrt{2}t} + C_2(1 + \sqrt{2})e^{\sqrt{2}t}. \end{aligned} \quad (2.6.45)$$

Thus, we get the same results as in direct method. The constants C_1 and C_2, evaluated using the boundary conditions (2.6.26) are the same as given in (2.6.34).

The solution for the set of differential equations (2.6.32) with the boundary conditions (2.6.26) for Example 2.11 using Symbolic Toolbox of the MATLAB©, Version 6, is shown below.

```
*******************************************************
x=dsolve('D2x-2*x=0','x(0)=1,x(1)=0')

x =

-(exp(2^(1/2))^2+1)/(exp(2^(1/2))^2-1)*sinh(2^(1/2)*t)+
cosh(2^(1/2)*t)

          1/2 2              1/2
    (exp(2    )  + 1) sinh(2     t)           1/2
  - ----------------------------- + cosh(2     t)
         1/2 2
    exp(2    )  - 1

u =

-(exp(2^(1/2))^2+1)/(exp(2^(1/2))^2-1)*cosh(2^(1/2)*t)*2^(1/2)+
sinh(2^(1/2)*t)*2^(1/2)-(exp(2^(1/2))^2+1)/(exp(2^(1/2))^2-
1)*sinh(2^(1/2)*t)+cosh(2^(1/2)*t)

          1/2 2              1/2      1/2
    (exp(2    )  + 1) cosh(2     t) 2             1/2      1/2
  - ---------------------------------- + sinh(2     t) 2
                   1/2 2
              exp(2    )  - 1

                 1/2 2              1/2
           (exp(2    )  + 1) sinh(2     t)           1/2
         - ----------------------------- + cosh(2     t)
                        1/2 2
                   exp(2    )  - 1
*******************************************************
```

It is easy to see that the previous solution for optimal $x^*(t)$ is the same as given in (2.6.33) and (2.6.34).

Let us note once again that the Lagrange multiplier $\lambda(t)$ helped us to treat the augmented functional (2.6.38) as if it contained *independent* variables $x(t)$ and $u(t)$, although they are *dependent* as per the plant equation (2.6.36).

2.7 Variational Approach to Optimal Control Systems

In this section, we approach the optimal control system by variational techniques, and in the process introduce the Hamiltonian function, which was used by Pontryagin and his associates to develop the famous *Minimum Principle* [109].

2.7.1 Terminal Cost Problem

Here we consider the optimal control system where the performance index is of general form containing a final (terminal) cost function in addition to the integral cost function. Such an optimal control problem is called the *Bolza* problem. Consider the plant as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{2.7.1}$$

the performance index as

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.7.2}$$

and given boundary conditions as

$$\mathbf{x}(t_0) = \mathbf{x}_0; \quad \mathbf{x}(t_f) \text{ is free and } t_f \text{ is free} \tag{2.7.3}$$

where, $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are n- and r- dimensional state and control vectors respectively. This *problem of Bolza* is the one with the most general form of the performance index.

> *The Lagrange problem was first discussed in 1762, Mayer considered his problem in 1878, and the problem of Bolza was formulated in 1913.*

Before we begin illustrating the Pontryagin procedure for this problem, let us note that

$$\int_{t_0}^{t_f} \frac{dS(\mathbf{x}(t), t)}{dt} dt = S(\mathbf{x}(t), t)|_{t_0}^{t_f} = S(\mathbf{x}(t_f), t_f) - S(\mathbf{x}(t_0), t_0). \tag{2.7.4}$$

Using the equation (2.7.4) in the original performance index (2.7.2), we get

$$\begin{aligned} J_2(\mathbf{u}(t)) &= \int_{t_0}^{t_f} \left[V(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{dS}{dt} \right] dt \\ &= \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt + S(\mathbf{x}(t_f), t_f) - S(\mathbf{x}(t_0), t_0). \end{aligned} \tag{2.7.5}$$

Since $S(\mathbf{x}(t_0), t_0)$ is a fixed quantity, the optimization of the original performance index J in (2.7.2) is equivalent to that of the performance index J_2 in (2.7.5). However, the *optimal cost* given by (2.7.2) is different from the optimal cost (2.7.5). Here, we are interested in finding the optimal control only. Once the optimal control is determined, the optimal cost is found using the original performance index J in (2.7.2) and not J_2 in (2.7.5). Also note that

$$\frac{d[S(\mathbf{x}(t), t)]}{dt} = \left(\frac{\partial S}{\partial \mathbf{x}}\right)' \dot{\mathbf{x}}(t) + \frac{\partial S}{\partial t}. \tag{2.7.6}$$

We now illustrate the procedure in the following steps. Also, we first introduce the Lagrangian and then, a little later, introduce the Hamiltonian. Let us first list the various steps and then describe the same in detail.

- **Step 1:** *Assumption of Optimal Conditions*
- **Step 2:** *Variations of Control and State Vectors*
- **Step 3:** *Lagrange Multiplier*
- **Step 4:** *Lagrangian*
- **Step 5:** *First Variation*
- **Step 6:** *Condition for Extrema*
- **Step 7:** *Hamiltonian*

- **Step 1:** *Assumptions of Optimal Conditions:* We assume optimum values $\mathbf{x}^*(t)$ and $\mathbf{u}^*(t)$ for state and control, respectively. Then

$$J(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{dS(\mathbf{x}^*(t), t)}{dt}\right] dt$$
$$\dot{\mathbf{x}}^*(\mathbf{t}) = \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t). \tag{2.7.7}$$

- **Step 2:** *Variations of Controls and States:* We consider the variations (perturbations) in control and state vectors as (see Figure 2.7)

$$\mathbf{x}(t) = \mathbf{x}^*(t) + \delta\mathbf{x}(t); \quad \mathbf{u}(t) = \mathbf{u}^*(t) + \delta\mathbf{u}(t). \tag{2.7.8}$$

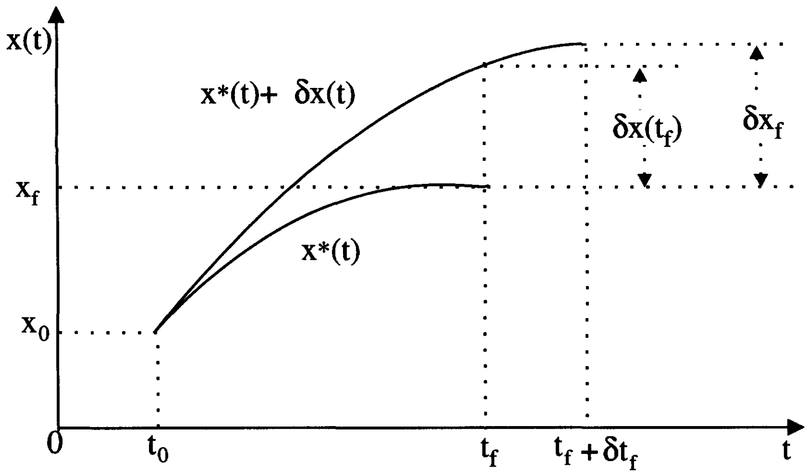

Figure 2.7 Free-Final Time and Free-Final State System

Then, the state equation (2.7.1) and the performance index (2.7.5) become

$$\dot{\mathbf{x}}^*(\mathbf{t}) + \delta\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{f}(\mathbf{x}^*(\mathbf{t}) + \delta\mathbf{x}(\mathbf{t}), \mathbf{u}^*(\mathbf{t}) + \delta\mathbf{u}(\mathbf{t}), \mathbf{t})$$
$$J(\mathbf{u}(t)) = \int_{t_0}^{t_f+\delta t_f} \left[V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) + \frac{dS}{dt}\right] dt \tag{2.7.9}$$

- **Step 3:** *Lagrange Multiplier:* Introducing the Lagrange multiplier vector $\boldsymbol{\lambda}(t)$ (also called costate vector) and using (2.7.6), we introduce the augmented performance index at the optimal condition as

$$J_a(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} [V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_*$$
$$+\boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}]dt \tag{2.7.10}$$

and at any other (perturbed) condition as

$$
\begin{aligned}
J_a(\mathbf{u}(t)) = \int_{t_0}^{t_f+\delta t_f} & [V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) \\
& + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' [\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)] + \left(\frac{\partial S}{\partial t}\right)_* \\
& + \boldsymbol{\lambda}'(t) \left[\mathbf{f}(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t)\right. \\
& \left. - \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\right]]dt. \qquad (2.7.11)
\end{aligned}
$$

- **Step 4:** *Lagrangian:* Let us define the Lagrangian function at optimal condition as

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\
&= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\
&\quad + \boldsymbol{\lambda}'(t) \{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\} \qquad (2.7.12)
\end{aligned}
$$

  and at any other condition as

$$
\begin{aligned}
\mathcal{L}^\delta &= \mathcal{L}^\delta(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\
&= V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) \\
&\quad + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' [\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)] + \left(\frac{\partial S}{\partial t}\right)_* \\
&\quad + \boldsymbol{\lambda}'(t) \left[\mathbf{f}(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t)\right. \\
&\quad \left. - \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\right]. \qquad (2.7.13)
\end{aligned}
$$

  With these, the augmented performance index at the optimal and any other condition becomes

$$
\begin{aligned}
J_a(\mathbf{u}^*(t)) &= \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t)dt = \int_{t_0}^{t_f} \mathcal{L}dt \\
J_a(\mathbf{u}(t)) &= \int_{t_0}^{t_f+\delta t_f} \mathcal{L}^\delta dt = \int_{t_0}^{t_f} \mathcal{L}^\delta dt + \int_{t_f}^{t_f+\delta t_f} \mathcal{L}^\delta dt. \qquad (2.7.14)
\end{aligned}
$$

Using mean-value theorem and Taylor series, and retaining the *linear* terms only, we have

$$\begin{aligned}\int_{t_f}^{t_f+\delta t_f} \mathcal{L}^\delta dt &= \mathcal{L}^\delta\Big|_{t_f} \delta t_f \\ &\approx \left\{\mathcal{L} + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*' \delta \mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) \right. \\ &\quad \left. + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta \mathbf{u}(t)\right\}\Big|_{t_f} \delta t_f \\ &\approx \mathcal{L}|_{t_f} \delta t_f. \end{aligned} \tag{2.7.15}$$

- **Step 5:** *First Variation:* Defining increment ΔJ, using Taylor series expansion, extracting the first variation δJ by retaining only the first order terms, we get the first variation as

$$\begin{aligned}\Delta J &= J_a(\mathbf{u}(t)) - J_a(\mathbf{u}^*(t)) \\ &= \int_{t_0}^{t_f} (\mathcal{L}^\delta - \mathcal{L}) dt + \mathcal{L}|_{t_f} \delta t_f \\ \delta J &= \int_{t_0}^{t_f} \left\{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*' \delta \mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta \mathbf{u}(t)\right\} dt \\ &\quad + \mathcal{L}|_{t_f} \delta t_f. \end{aligned} \tag{2.7.16}$$

Considering the $\delta\dot{\mathbf{x}}(t)$ term in the first variation (2.7.16) and integrating by parts (using $\int u dv = uv - \int v du$),

$$\begin{aligned}\int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) dt &= \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \frac{d}{dt} (\delta \mathbf{x}(t))\, dt \\ &= \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \mathbf{x}(t)\right]\Bigg|_{t_0}^{t_f} \\ &\quad - \int_{t_0}^{t_f} \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*\right]' \delta \mathbf{x}(t) dt. \end{aligned} \tag{2.7.17}$$

Also note that since $\mathbf{x}(t_0)$ is specified, $\delta\mathbf{x}(t_0) = 0$. Thus, using (2.7.17) the first variation δJ in (2.7.16) becomes

$$\delta J = \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* \right]' \delta\mathbf{x}(t)dt$$
$$+ \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta\mathbf{u}(t)dt$$
$$+ \mathcal{L}|_{t_f}\delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta\mathbf{x}(t) \right]\Bigg|_{t_f}. \tag{2.7.18}$$

- **Step 6:** *Condition for Extrema:* For extrema of the functional J, the *first variation* δJ should vanish according to the fundamental theorem (Theorem 2.1) of the CoV. Also, in a typical control system such as (2.7.1), we note that $\delta\mathbf{u}(t)$ is the *independent* control variation and $\delta\mathbf{x}(t)$ is the *dependent* state variation. First, we choose $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}^*(t)$ which is at our disposal and hence $\mathcal{L}^*$ such that the coefficient of the *dependent* variation $\delta\mathbf{x}(t)$ in (2.7.18) be zero. Then, we have the Euler-Lagrange equation

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0 \tag{2.7.19}$$

where the partials are evaluated at the optimal (*) condition. Next, since the independent control variation $\delta\mathbf{u}(t)$ is arbitrary, the coefficient of the control variation $\delta\mathbf{u}(t)$ in (2.7.18) should be set to zero. That is

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0. \tag{2.7.20}$$

Finally, the first variation (2.7.18) reduces to

$$\mathcal{L}^*|_{t_f}\delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta\mathbf{x}(t) \right]\Bigg|_{t_f} = 0. \tag{2.7.21}$$

Let us note that the condition (or plant) equation (2.7.1) can be written in terms of the Lagrangian (2.7.12) as

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* = 0. \tag{2.7.22}$$

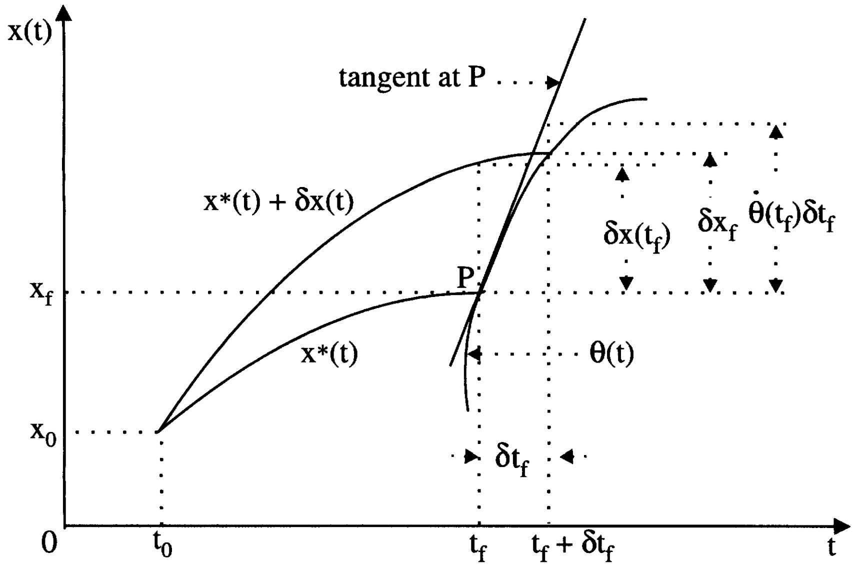

Figure 2.8 Final-Point Condition with a Moving Boundary $\theta(t)$

In order to convert the expression containing $\delta \mathbf{x}(t)$ in (2.7.21) into an expression containing $\delta \mathbf{x}_f$ (see Figure 2.8), we note that the slope of $\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)$ at t_f is approximated as

$$\dot{\mathbf{x}}^*(t_f) + \delta\dot{\mathbf{x}}(t_f) \approx \frac{\delta \mathbf{x}_f - \delta \mathbf{x}(t_f)}{\delta t_f} \tag{2.7.23}$$

which is rewritten as

$$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f) + \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\,\delta t_f \tag{2.7.24}$$

and retaining only the linear (in δ) terms in the relation (2.7.24), we have

$$\delta \mathbf{x}(t_f) = \delta \mathbf{x}_f - \dot{\mathbf{x}}^*(t_f)\delta t_f. \tag{2.7.25}$$

Using (2.7.25) in the boundary condition (2.7.21), we have the general boundary condition in terms of the Lagrangian as

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_* \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_*\Bigg|_{t_f} \delta \mathbf{x}_f = 0. \tag{2.7.26}$$

- **Step 7:** *Hamiltonian:* We define the Hamiltonian $\mathcal{H}^*$ (also called the Pontryagin $\mathcal{H}$ function) at the optimal condition as

$$\boxed{\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*'}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t),} \tag{2.7.27}$$

where,

$$\mathcal{H}^* = \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t).$$

Then from (2.7.12) the Lagrangian $\mathcal{L}^*$ in terms of the Hamiltonian $\mathcal{H}^*$ becomes

$$\begin{aligned}\mathcal{L}^* &= \mathcal{L}^*(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &\quad + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_* - \boldsymbol{\lambda}^{*'}(t)\dot{\mathbf{x}}^*(t). \end{aligned} \tag{2.7.28}$$

Using (2.7.28) in (2.7.20), (2.7.19), and (2.7.22) and noting that the terminal cost function $S = S(\mathbf{x}(t), t)$, we have the control, state and costate equations, respectively expressed in terms of the Hamiltonian. Thus, for the optimal control $\mathbf{u}^*(t)$, the relation (2.7.20) becomes

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0 \longrightarrow \boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0} \tag{2.7.29}$$

for the optimal state $\mathbf{x}^*(t)$, the relation (2.7.19) becomes

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}^*}{\partial \dot{\mathbf{x}}}\right)_* = 0 \longrightarrow$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* + \left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x} \partial t}\right)_* - \frac{d}{dt}\left\{\left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' - \boldsymbol{\lambda}^*(t)\right\} = 0 \longrightarrow$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* + \left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x} \partial t}\right)_* - \left[\left(\frac{\partial^2 S}{\partial \mathbf{x}^2}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial^2 S}{\partial \mathbf{x} \partial t}\right)_* - \dot{\boldsymbol{\lambda}}^*(t)\right] = 0$$

leading to

$$\boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)} \tag{2.7.30}$$

and for the costate $\boldsymbol{\lambda}^*(t)$,

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* = 0 \longrightarrow \boxed{\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).} \tag{2.7.31}$$

Looking at the similar structure of the relation (2.7.30) for the optimal costate $\boldsymbol{\lambda}^*(t)$ and (2.7.31) for the optimal state $\mathbf{x}^*(t)$ it is clear why $\boldsymbol{\lambda}(t)$ is called the *costate* vector. Finally, using the relation (2.7.28), the boundary condition (2.7.26) at the optimal condition reduces to

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.} \tag{2.7.32}$$

This is the general boundary condition for free-end point system in terms of the Hamiltonian.

2.7.2 Different Types of Systems

We now obtain different cases depending on the statement of the problem regarding the final time t_f and the final state $\mathbf{x}(t_f)$ (see Figure 2.9).

- **Type (a):** *Fixed-Final Time and Fixed-Final State System:* Here, since t_f and $\mathbf{x}(t_f)$ are fixed or specified (Figure 2.9(a)), both δt_f and $\delta \mathbf{x}_f$ are zero in the general boundary condition (2.7.32), and there is no extra boundary condition to be used other than those given in the problem formulation.

- **Type (b):** *Free-Final Time and Fixed-Final State System:* Since t_f is free or not specified in advance, δt_f is arbitrary, and since $\mathbf{x}(\mathbf{t_f})$ is fixed or specified, $\delta \mathbf{x}_f$ is zero as shown in Figure 2.9(b). Then, the coefficient of the arbitrary δt_f in the general boundary condition (2.7.32) is zero resulting in

$$\boxed{\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{*t_f} = 0.} \tag{2.7.33}$$

- **Type (c):** *Fixed-Final Time and Free-Final State System:* Here t_f is specified and $\mathbf{x}(t_f)$ is free (see Figure 2.9(c)). Then δt_f is zero and $\delta \mathbf{x}_f$ is arbitrary, which in turn means that the coefficient of $\delta \mathbf{x}_f$ in the general boundary condition (2.7.32) is zero. That is

$$\left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_{*t_f} = 0 \longrightarrow \boxed{\boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}.} \tag{2.7.34}$$

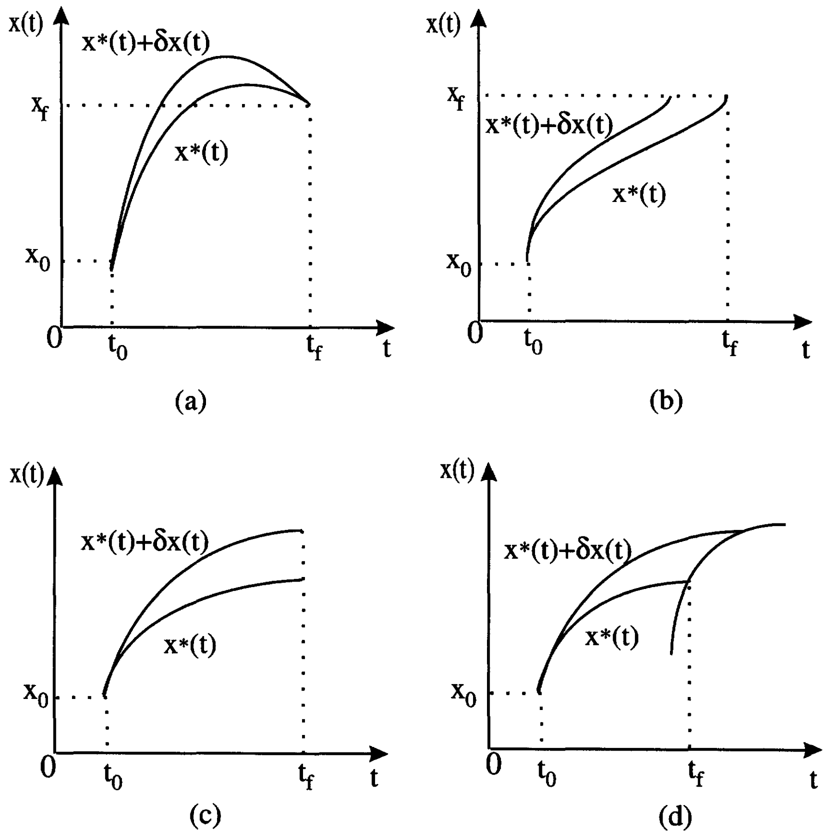

Figure 2.9 Different Types of Systems: (a) Fixed-Final Time and Fixed-Final State System, (b) Free-Final Time and Fixed-Final State System, (c) Fixed-Final Time and Free-Final State System, (d) Free-Final Time and Free-Final State System

- **Type (d):** *Free-Final Time and Dependent Free-Final State System:* If t_f and $\mathbf{x}(t_f)$ are *related* such that $\mathbf{x}(t_f)$ lies on a moving curve $\boldsymbol{\theta}(t)$ as shown in Figure 2.8, then

$$\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f) \quad \text{and} \quad \delta\mathbf{x}_f \approx \dot{\boldsymbol{\theta}}(t_f)\delta t_f. \tag{2.7.35}$$

Using (2.7.35), the boundary condition (2.7.32) for the optimal condition becomes

$$\left[\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_* + \left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_*' \dot{\boldsymbol{\theta}}(t)\right]_{t_f} \delta t_f = 0. \tag{2.7.36}$$

Since t_f is free, δt_f is arbitrary and hence the coefficient of δt_f in (2.7.36) is zero. That is

$$\boxed{\left[\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_* + \left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_*' \dot{\boldsymbol{\theta}}(t)\right]_{t_f} = 0.} \tag{2.7.37}$$

- **Type (e):** *Free-Final Time and Independent Free-Final State:* If t_f and $\mathbf{x}(t_f)$ are *not related*, then δt_f and $\delta\mathbf{x}_f$ are unrelated, and the boundary condition (2.7.32) at the optimal condition becomes

$$\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{*t_f} = 0 \tag{2.7.38}$$

$$\left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_{*t_f} = 0. \tag{2.7.39}$$

2.7.3 Sufficient Condition

In order to determine the nature of optimization, i.e., whether it is minimum or maximum, we need to consider the second variation and examine its sign. In other words, we have to find a *sufficient* condition for extremum. Using (2.7.14), (2.7.28) and (2.7.37), we have the second variation in (2.7.16) and using the relation (2.7.28), we get

$$\begin{aligned}\delta^2 J &= \int_{t_0}^{t_f} \left[\frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2}(\delta \mathbf{x}(t))^2 + \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}(\delta \mathbf{u}(t))^2 + 2\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}\partial \mathbf{x}}(\delta \mathbf{u}(t)\delta \mathbf{x}(t)) \right]_* dt \\ &= \int_{t_0}^{t_f} \left[\delta \mathbf{x}'(t) \;\; \delta \mathbf{u}'(t) \right] \begin{bmatrix} \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}\partial \mathbf{u}} \\ \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}\partial \mathbf{u}} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \end{bmatrix}_* \begin{bmatrix} \delta \mathbf{x}(t) \\ \delta \mathbf{u}(t) \end{bmatrix} dt \\ &= \int_{t_0}^{t_f} \left[\delta \mathbf{x}'(t) \;\; \delta \mathbf{u}'(t) \right] \Pi \begin{bmatrix} \delta \mathbf{x}(t) \\ \delta \mathbf{u}(t) \end{bmatrix} dt. \end{aligned} \tag{2.7.40}$$

For the *minimum*, the second variation $\delta^2 J$ must be *positive*. This means that the matrix Π in (2.7.40)

$$\Pi = \begin{bmatrix} \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}\partial \mathbf{u}} \\ \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}\partial \mathbf{u}} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \end{bmatrix}_* \tag{2.7.41}$$

must be *positive definite*. But the important condition is that the second partial derivative of $\mathcal{H}^*$ w.r.t. $\mathbf{u}(t)$ must be positive. That is

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \right)_* > 0} \tag{2.7.42}$$

and for the *maximum*, the sign of (2.7.42) is reversed.

2.7.4 Summary of Pontryagin Procedure

Consider a free-final time and free-final state problem with general cost function (Bolza problem), where we want to minimize the performance index

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.7.43}$$

for the plant described by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.7.44}$$

with the boundary conditions as

$$\mathbf{x}(t = t_0) = \mathbf{x}_0; \quad t = t_f \text{ is free and } \mathbf{x}(t_f) \text{ is free.} \tag{2.7.45}$$

Table 2.1 Procedure Summary of Pontryagin Principle for Bolza Problem

A. Statement of the Problem
Given the plant as $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$,   the performance index as $J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$,   and the boundary conditions as $\mathbf{x}(t_0) = \mathbf{x}_0$ and t_f and $\mathbf{x}(t_f) = \mathbf{x}_f$ are free,   find the optimal control.

B. Solution of the Problem	
Step 1	Form the Pontryagin $\mathcal{H}$ function $\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$.
Step 2	Minimize $\mathcal{H}$ w.r.t. $\mathbf{u}(t)$ $\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$ and obtain $\mathbf{u}^*(t) = \mathbf{h}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t)$.
Step 3	Using the results of Step 2 in Step 1, find the optimal $\mathcal{H}^*$ $\mathcal{H}^*(\mathbf{x}^*(t), \mathbf{h}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t), \boldsymbol{\lambda}^*(t), t) = \mathcal{H}^*(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t)$.
Step 4	Solve the set of 2n differential equations $\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*$ and $\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*$ with initial conditions $\mathbf{x}_0$ and the final conditions $\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0$.
Step 5	Substitute the solutions of $\mathbf{x}^*(t)$, $\boldsymbol{\lambda}^*(t)$ from Step 4 into the expression for the optimal control $\mathbf{u}^*(t)$ of Step 2.

C. Types of Systems
(a). Fixed-final time and fixed-final state system, Fig. 2.9(a)
(b). Free-final time and fixed-final state system, Fig. 2.9(b)
(c). Fixed-final time and free-final state system, Fig. 2.9(c)
(d). Free-final time and dependent free-final state system, Fig. 2.9(d).
(e). Free-final time and independent free-final state system

Type	Substitutions	Boundary Conditions
(a)	$\delta t_f = 0, \delta \mathbf{x}_f = 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \mathbf{x}_f$
(b)	$\delta t_f \neq 0, \delta \mathbf{x}_f = 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \mathbf{x}_f, \left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0$
(c)	$\delta t_f = 0, \delta \mathbf{x}_f \neq 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}$
(d)	$\delta \mathbf{x}_f = \dot{\boldsymbol{\theta}}(t_f)\delta t_f$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ $\left[\mathcal{H}^* + \frac{\partial S}{\partial t} + \left\{\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right\}' \dot{\boldsymbol{\theta}}(t)\right]_{t_f} = 0$
(e)	$\delta t_f \neq 0$ $\delta \mathbf{x}_f \neq 0$	$\delta \mathbf{x}(t_0) = \mathbf{x}_0$ $\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0, \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f} = 0$

Here, $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are $n-$ and $r-$ dimensional state and control vectors respectively. Let us note that $\mathbf{u}(t)$ is *unconstrained.* The entire procedure (called Pontryagin Principle) is now summarized in Table 2.1.

Note: From Table 2.1 we note that the only difference in the procedure between the *free-final point system without the final cost function* (Lagrange problem) and *free-final point system with final cost function* (Bolza problem) is in the application of the general boundary condition.

To illustrate the Pontryagin method described previously, consider the following simple examples describing a second order system. Specifically, we selected a double integral plant whose analytical solutions for the optimal condition can be obtained and the same verified by using MATLAB©.

First we consider the fixed-final time and fixed-final state problem (Figure 2.9(a), Table 2.1, Type (a)).

Example 2.12

Given a second order (double integrator) system as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u(t)\end{aligned} \tag{2.7.46}$$

and the performance index as

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt \tag{2.7.47}$$

find the optimal control and optimal state, given the boundary (initial and final) conditions as

$$\mathbf{x}(0) = [1 \;\; 2]'; \qquad \mathbf{x}(2) = [1 \;\; 0]'. \tag{2.7.48}$$

Assume that the control and state are unconstrained.

Solution: We follow the step-by-step procedure given in Table 2.1. First, by comparing the present plant (2.7.46) and the PI (2.7.47) with the general formulation of the plant (2.7.1) and the PI (2.7.2), we identify

$$\begin{aligned}V(\mathbf{x}(t),\mathbf{u}(t),t) &= V(u(t)) = \frac{1}{2}u^2(t)\\ \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t) &= [f_1,\, f_2]',\end{aligned} \tag{2.7.49}$$

where, $f_1 = x_2(t), \quad f_2 = u(t)$.

- **Step 1:** Form the Hamiltonian function as

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) \\ &= V(u(t)) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ &= \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t). \end{aligned} \quad (2.7.50)$$

- **Step 2:** Find $u^*(t)$ from

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow u^*(t) + \lambda_2^*(t) = 0 \longrightarrow u^*(t) = -\lambda_2^*(t). \quad (2.7.51)$$

- **Step 3:** Using the results of Step 2 in Step 1, find the optimal $\mathcal{H}^*$ as

$$\begin{aligned}\mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) &= \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t) \\ &= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t). \end{aligned} \quad (2.7.52)$$

- **Step 4:** Obtain the state and costate equations from

$$\begin{aligned}\dot{x}_1^*(t) &= +\left(\frac{\partial \mathcal{H}}{\partial \lambda_1}\right)_* = x_2^*(t) \\ \dot{x}_2^*(t) &= +\left(\frac{\partial \mathcal{H}}{\partial \lambda_2}\right)_* = -\lambda_2^*(t) \\ \dot{\lambda}_1^*(t) &= -\left(\frac{\partial \mathcal{H}}{\partial x_1}\right)_* = 0 \\ \dot{\lambda}_2^*(t) &= -\left(\frac{\partial \mathcal{H}}{\partial x_2}\right)_* = -\lambda_1^*(t). \end{aligned} \quad (2.7.53)$$

  Solving the previous equations, we have the optimal state and costate as

$$\begin{aligned}x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1 \\ x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2 \\ \lambda_1^*(t) &= C_3 \\ \lambda_2^*(t) &= -C_3 t + C_4. \end{aligned} \quad (2.7.54)$$

- **Step 5:** Obtain the optimal control from

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4 \quad (2.7.55)$$

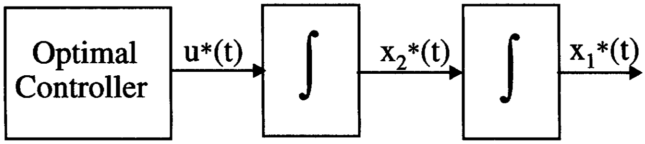

Figure 2.10 Optimal Controller for Example 2.12

where, C_1, C_2, C_3, and C_4 are constants evaluated using the given boundary conditions (2.7.48). These are found to be

$$C_1 = 1, \quad C_2 = 2, \quad C_3 = 3, \quad \text{and} \quad C_4 = 4. \tag{2.7.56}$$

Finally, we have the optimal states, costates and control as

$$\begin{aligned} x_1^*(t) &= 0.5t^3 - 2t^2 + 2t + 1, \\ x_2^*(t) &= 1.5t^2 - 4t + 2, \\ \lambda_1^*(t) &= 3, \\ \lambda_2^*(t) &= -3t + 4, \\ u^*(t) &= 3t - 4. \end{aligned} \tag{2.7.57}$$

The system with the optimal controller is shown in Figure 2.10.

The solution for the set of differential equations (2.7.53) with the boundary conditions (2.7.48) for Example 2.12 using Symbolic Toolbox of the MATLAB©, Version 6, is shown below.

```
*************************************************************
%% Solution Using Symbolic Toolbox (STB) in
%% MATLAB Version 6.0
%%
S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1,...
x1(0)=1,x2(0)=2,x1(2)=1,x2(2)=0')
S.x1
S.x2
S.lambda1
S.lambda2

S =

    lambda1: [1x1 sym]
    lambda2: [1x1 sym]
         x1: [1x1 sym]
         x2: [1x1 sym]
S.x1

ans =
```

```
1+2*t-2*t^2+1/2*t^3

S.x2

ans =
2-4*t+3/2*t^2

S.lambda1

ans =
3
S.lambda2

ans =

4-3*t

Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
*********************************************************
```

It is easy to see that the previous solutions for $x_1^*(t), x_2^*(t), \lambda_1^*(t)$, $\lambda_2^*(t)$, and $u^*(t) = -\lambda_2^*(t)$ obtained by using MATLAB© are the same as those given by the analytical solutions (2.7.57). The optimal control and state are plotted (using MATLAB©) in Figure 2.11.

Next, we consider the fixed-final time and free-final state case (Fig-

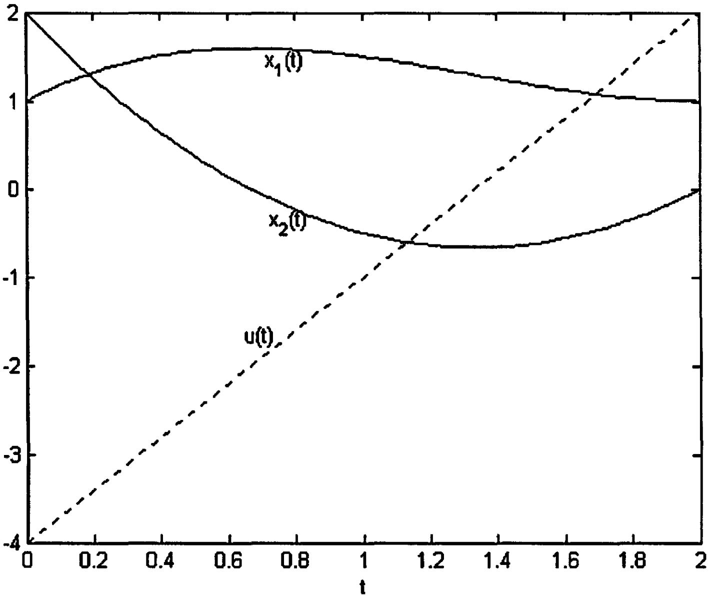

Figure 2.11 Optimal Control and States for Example 2.12

ure 2.9(b), Table 2.1, Type (c)) of the same system.

Example 2.13

Consider the same Example 2.12 with changed boundary conditions as

$$\mathbf{x}(0) = [1 \;\; 2]'; \qquad x_1(2) = 0; \qquad x_2(2) \text{ is free.} \tag{2.7.58}$$

Find the optimal control and optimal states.

Solution: Following the procedure illustrated in Table 2.1 (Type (c)), we get the same optimal states, costates, and control as given in (2.7.54) and (2.7.55) which are repeated here for convenience.

$$\begin{aligned}
x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1, \\
x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2, \\
\lambda_1^*(t) &= C_3, \\
\lambda_2^*(t) &= -C_3 t + C_4, \\
u^*(t) &= -\lambda_2^*(t) = C_3 t - C_4.
\end{aligned} \tag{2.7.59}$$

The only difference is in solving for the constants C_1 to C_4. First of all, note that the performance index (2.7.47) does not contain the terminal cost function S. From the given boundary conditions

(2.7.58), we have t_f specified to be 2 and hence δt_f is zero in the general boundary condition (2.7.32).

Also, since $x_2(2)$ is free, δx_{2_f} is arbitrary and hence the corresponding final condition on the costate becomes

$$\lambda_2(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{*t_f} = 0 \tag{2.7.60}$$

(since $S = 0$). Thus we have the four boundary conditions as

$$x_1(0) = 1; \quad x_2(0) = 2; \quad x_1(2) = 0; \quad \lambda_2(2) = 0. \tag{2.7.61}$$

With these boundary conditions substituted in (2.7.59), the constants are found to be

$$C_1 = 1; \quad C_2 = 2; \quad C_3 = 15/8; \quad C_4 = 15/4. \tag{2.7.62}$$

Finally the optimal states, costates and control are given from (2.7.59) and (2.7.62) as

$$\begin{aligned}
x_1^*(t) &= \frac{5}{16}t^3 - \frac{15}{8}t^2 + 2t + 1, \\
x_2^*(t) &= \frac{15}{16}t^2 - \frac{15}{4}t + 2, \\
\lambda_1^*(t) &= \frac{15}{8}, \\
\lambda_2^*(t) &= -\frac{15}{8}t + \frac{15}{4}, \\
u^*(t) &= \frac{15}{8}t - \frac{15}{4}.
\end{aligned} \tag{2.7.63}$$

The solution for the set of differential equations (2.7.53) with the boundary conditions (2.7.58) for Example 2.13 using Symbolic Toolbox of the MATLAB©, Version 6, is shown below.

```
*********************************************************************
%% Solution Using Symbolic Toolbox (STB) in
%% MATLAB Version 6.0
%%
S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1,
x1(0)=1,x2(0)=2,x1(2)=0,lambda2(2)=0')

S =
```

```
    lambda1: [1x1 sym]
    lambda2: [1x1 sym]
         x1: [1x1 sym]
         x2: [1x1 sym]
S.x1

ans =

5/16*t^3+2*t+1-15/8*t^2

S.x2

ans =

15/16*t^2+2-15/4*t

S.lambda1

ans =

15/8

S.lambda2

ans =

-15/8*t+15/4

%% Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
```

```
gtext('x_2(t)')
gtext('u(t)')
**************************************************
```

It is easy to see that the previous solutions for $x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)$, and $u^*(t) = -\lambda_2^*(t)$ obtained by using MATLAB© are the same as those given by (2.7.63) obtained analytically. The optimal control and states for Example 2.13 are plotted in Figure 2.12.

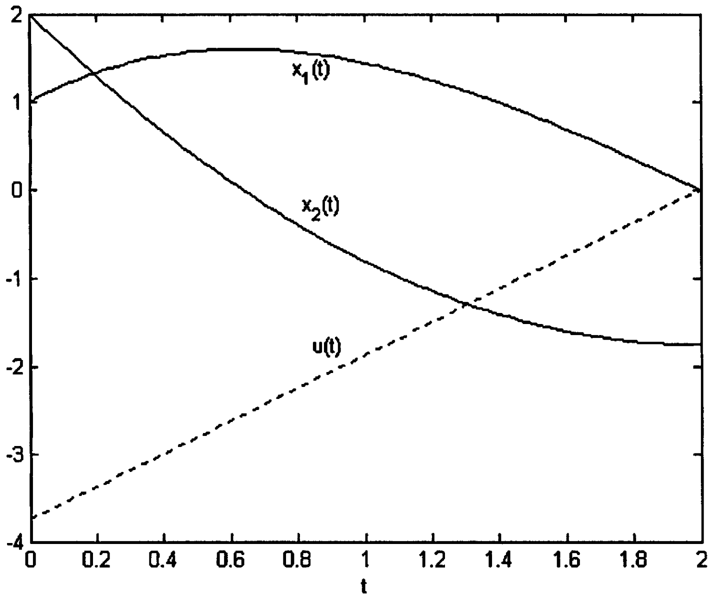

Figure 2.12 Optimal Control and States for Example 2.13

Next, we consider the free-final time and independent free-final state case (Figure 2.9(e), Table 2.1, Type (e)) of the same system.

Example 2.14

Consider the same Example 2.12 with changed boundary conditions as

$$\mathbf{x}(0) = [1 \;\; 2]'; \quad x_1(2) = 0; \quad x_1(t_f) = 3; \quad x_2(t_f) \text{ is free.} \quad (2.7.64)$$

Find the optimal control and optimal state.

Solution: Following the procedure illustrated in Table 2.1 (Type (e)), we get the same optimal control, states and costates as given in (2.7.54) and (2.7.55) which are repeated here for convenience.

$$
\begin{aligned}
x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1, \\
x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2, \\
\lambda_1^*(t) &= C_3, \\
\lambda_2^*(t) &= -C_3 t + C_4, \\
u^*(t) &= -\lambda_2^*(t) = C_3 t - C_4.
\end{aligned}
\tag{2.7.65}
$$

The only difference is in solving for the constants C_1 to C_4 and the unknown t_f. First of all, note that the performance index (2.7.47) does not contain the terminal cost function S, that is, $S = 0$. From the given boundary conditions (2.7.64), we have t_f unspecified and hence δt_f is free in the general boundary condition (2.7.32) leading to the specific final condition

$$\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{t_f} = 0 \longrightarrow \lambda_1(t_f)x_2(t_f) - 0.5\lambda_2^2(t_f) = 0 \tag{2.7.66}$$

Also, since $x_2(t_f)$ is free, δx_{2_f} is arbitrary and hence the general boundary condition (2.7.32) becomes

$$\lambda_2(t_f) = \left(\frac{\partial S}{\partial x_2}\right) = 0 \tag{2.7.67}$$

where $\mathcal{H}$ is given by (2.7.52). Combining (2.7.64), (2.7.66) and (2.7.67), we have the following 5 boundary conditions for the 5 unknowns (4 constants of integration C_1 to C_4 and 1 unknown t_f) as

$$
\begin{gathered}
x_1(0) = 1; \quad x_2(0) = 2; \quad x_1(t_f) = 3; \\
\lambda_2(t_f) = 0; \quad \lambda_1(t_f)x_2(t_f) - 0.5\lambda_2^2(t_f) = 0.
\end{gathered}
\tag{2.7.68}
$$

Using these boundary conditions along with (2.7.65), the constants are found to be

$$C_1 = 1; \quad C_2 = 2; \quad C_3 = 4/9; \quad C_4 = 4/3; \quad t_f = 3. \tag{2.7.69}$$

Finally, the optimal states, costates, and control are given from

(2.7.65) and (2.7.69) as

$$\begin{aligned} x_1^*(t) &= \frac{4}{54}t^3 - \frac{2}{3}t^2 + 2t + 1, \\ x_2^*(t) &= \frac{4}{18}t^2 - \frac{4}{3}t + 2, \\ \lambda_1^*(t) &= \frac{4}{9}, \\ \lambda_2^*(t) &= -\frac{4}{9}t + \frac{4}{3}, \\ u^*(t) &= \frac{4}{9}t - \frac{4}{3}. \end{aligned} \tag{2.7.70}$$

The solution for the set of differential equations (2.7.53) with the boundary conditions (2.7.68) for Example 2.14 using Symbolic Toolbox of the MATLAB©, Version 6 is shown below.

```
*********************************************************
%% Solution Using Symbolic Toolbox (STB) in
%% of MATLAB Version 6
%%
clear all
S=dsolve('Dx1=x2,Dx2=-lam2,Dlam1=0,Dlam2=-lam1,x1(0)=1,
    x2(0)=2,x1(tf)=3,lam2(tf)=0')
t='tf';
eq1=subs(S.x1)-'x1tf';
eq2=subs(S.x2)-'x2tf';
eq3=S.lam1-'lam1tf';
eq4=subs(S.lam2)-'lam2tf';
eq5='lam1tf*x2tf-0.5*lam2tf^2';
S2=solve(eq1,eq2,eq3,eq4,eq5,'tf,x1tf,x2tf,lam1tf,
    lam2tf','lam1tf<>0')
%% lam1tf<>0 means lam1tf is not equal to 0;
%% This is a condition derived from eq5.
%% Otherwise, without this condition in the above
%% SOLVE routine, we get two values for tf (1 and 3 in this case)
%%
tf=S2.tf
x1tf=S2.x1tf;
x2tf=S2.x2tf;
clear t
x1=subs(S.x1)
x2=subs(S.x2)
lam1=subs(S.lam1)
```

```
lam2=subs(S.lam2)
%% Convert the symbolic values to
%% numerical values as shown below.
j=1;
tf=double(subs(S2.tf))
%% coverts tf from symbolic to numerical
for tp=0:0.05:tf
t=sym(tp);
%% coverts tp from numerical to symbolic
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lam2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
*******************************************************
```

The optimal control and states for Example 2.14 are plotted in Figure 2.13.

Finally, we consider the fixed-final time and free-final state system with a terminal cost (Figure 2.9 (b), Table 2.1, Type (b)).

Example 2.15

We consider the same Example 2.12 with changed performance index

$$J = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2 + \frac{1}{2}\int_0^2 u^2 dt \qquad (2.7.71)$$

and boundary conditions as

$$\mathbf{x}(0) = [1 \;\; 2]; \qquad \mathbf{x}(2) = \text{is free.} \qquad (2.7.72)$$

Following the procedure illustrated in Table 2.1 (Type (b)), we get the same optimal control, states and costates as given in (2.7.54) and (2.7.55), which are reproduced here for ready reference. Thus

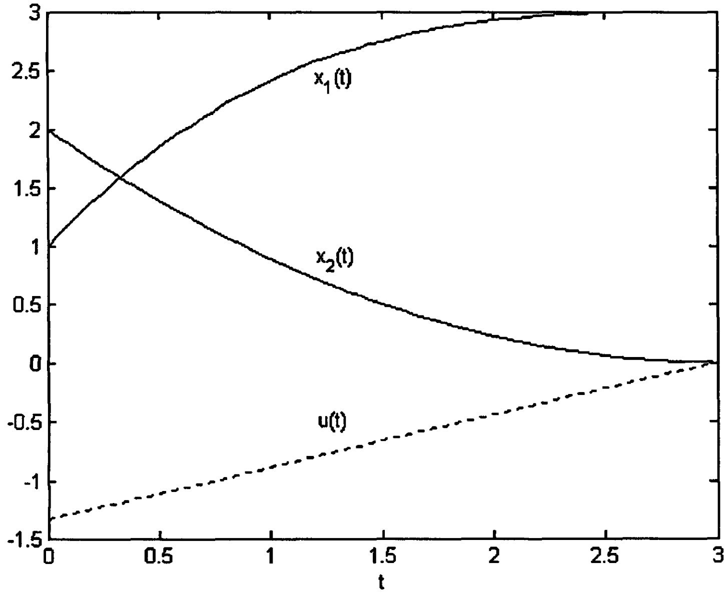

Figure 2.13 Optimal Control and States for Example 2.14

we have

$$\begin{aligned} x_1^*(t) &= \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1, \\ x_2^*(t) &= \frac{C_3}{2}t^2 - C_4 t + C_2, \\ \lambda_1^*(t) &= C_3, \\ \lambda_2^*(t) &= -C_3 t + C_4, \\ u^*(t) &= -\lambda_2(t) = C_3 t - C_4. \end{aligned} \tag{2.7.73}$$

The only difference is in solving for the constants C_1 to C_4 using the given and obtained boundary conditions. Since t_f is specified as 2, δt_f is zero and since $\mathbf{x}(2)$ unspecified, $\delta \mathbf{x}_f$ is free in the boundary condition (2.7.32), which now reduces to

$$\boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f} \tag{2.7.74}$$

where,

$$S(\mathbf{x}(t_f)) = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2. \tag{2.7.75}$$

Thus, (2.7.74) becomes

$$\lambda_1^*(t_f) = \left(\frac{\partial S}{\partial x_1}\right)_{t_f} \longrightarrow \lambda_1^*(2) = x_1(2) - 4$$
$$\lambda_2^*(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{t_f} \longrightarrow \lambda_2^*(2) = x_2(2) - 2. \tag{2.7.76}$$

Now, we have two initial conditions from (2.7.72) and two final conditions from (2.7.76) to solve for the four constants as

$$C_1 = 1, \quad C_2 = 2, \quad C_3 = \frac{3}{7}, \quad C_4 = \frac{4}{7}. \tag{2.7.77}$$

Finally, we have the optimal states, costates and control given as

$$\begin{aligned} x_1^*(t) &= \frac{1}{14}t^3 - \frac{2}{7}t^2 + 2t + 1, \\ x_2^*(t) &= \frac{3}{14}t^2 - \frac{4}{7}t + 2, \\ \lambda_1^*(t) &= \frac{3}{7}, \\ \lambda_2^*(t) &= -\frac{3}{7}t + \frac{4}{7}, \\ u^*(t) &= \frac{3}{7}t - \frac{4}{7}. \end{aligned} \tag{2.7.78}$$

The previous results can also obtained using Symbolic Math Toolbox of the MATLAB©, Version 6, as shown below.

```
*****************************************************************
%% Solution Using Symbolic Math Toolbox (STB) in
%%  MATLAB Version 6
%%
S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1,
x1(0)=1,x2(0)=2,lambda1(2)=x12-4,lambda2(2)=x22-2')
t='2';
S2=solve(subs(S.x1)-'x12',subs(S.x2)-'x22','x12,x22');
%% solves for x1(t=2) and x2(t=2)
x12=S2.x12;
x22=S2.x22;
clear t

S =
```

```
    lambda1: [1x1 sym]
    lambda2: [1x1 sym]
         x1: [1x1 sym]
         x2: [1x1 sym]

x1=subs(S.x1)

x1 =

1-2/7*t^2+1/14*t^3+2*t


x2=subs(S.x2)

x2 =

-4/7*t+3/14*t^2+2

lambda1=subs(S.lambda1)

lambda1 =

3/7


lambda2=subs(S.lambda2)

lambda2 =

4/7-3/7*t

%% Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
```

```
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
****************************************************************
```

It is easy to see that the previous solutions for $x_1^*(t)$, $x_2^*(t)$, $\lambda_1^*(t)$, $\lambda_2^*(t)$, and $u^*(t) = -\lambda_2^*(t)$ obtained by using MATLAB© are the same as those given by (2.7.78) obtained analytically.

The optimal control and states for Example 2.15 are plotted in Figure 2.14.

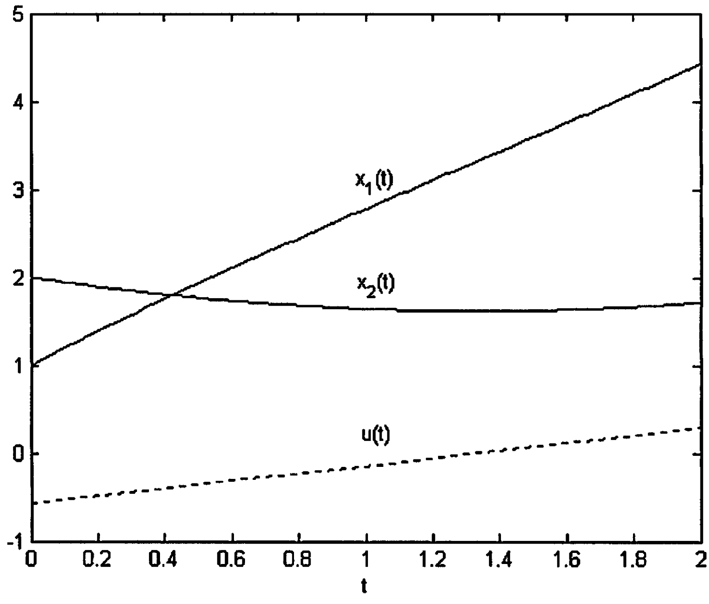

Figure 2.14 Optimal Control and States for Example 2.15

2.8 Summary of Variational Approach

In this section, we summarize the development of the topics covered so far in obtaining optimal conditions using the calculus of variations. The development is carried out in different stages as follows. Also shown is the systematic link between various stages of development.

2.8.1 Stage I: Optimization of a Functional

Consider the optimal of

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt \tag{2.8.1}$$

with the given boundary conditions

$$\mathbf{x}(t_0) \text{ fixed and } \mathbf{x}(t_f) \text{ free.} \tag{2.8.2}$$

Then, the optimal function $\mathbf{x}^*(t)$ should satisfy the Euler-Lagrange equation

$$\boxed{\left(\frac{\partial V}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{\mathbf{x}}}\right)_* = 0.} \tag{2.8.3}$$

The general boundary condition to be satisfied at the free-final point is given by [79]

$$\boxed{\left[V - \dot{\mathbf{x}}'(t)\left(\frac{\partial V}{\partial \dot{\mathbf{x}}}\right)\right]_{*t_f} \delta t_f + \left(\frac{\partial V}{\partial \dot{\mathbf{x}}}\right)'_{*t_f} \delta \mathbf{x}_f = 0.} \tag{2.8.4}$$

This boundary condition is to be modified depending on the nature of the given t_f and $\mathbf{x}(t_f)$. Although the previous general boundary condition is not derived in this book, it can be easily seen to be similar to the general boundary condition (2.7.26) in terms of the Lagrangian which embeds a performance index and a dynamical plant into a single augmented performance index with integrand $\mathcal{L}$.

The sufficient condition for optimum is the Legendre condition given by

$$\boxed{\left(\frac{\partial^2 V}{\partial \dot{\mathbf{x}}^2}\right)_* > 0} \quad \text{for minimum} \tag{2.8.5}$$

and

$$\boxed{\left(\frac{\partial^2 V}{\partial \dot{\mathbf{x}}^2}\right)_* < 0} \quad \text{for maximum.} \tag{2.8.6}$$

2.8.2 Stage II: Optimization of a Functional with Condition

Consider the optimization of a functional

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt \tag{2.8.7}$$

with given boundary values as

$$\mathbf{x}(t_0) \text{ fixed and } \mathbf{x}(t_f) \text{ free,} \tag{2.8.8}$$

and the condition relation

$$\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0. \tag{2.8.9}$$

Here, the condition (2.8.9) is absorbed by forming the augmented functional

$$J_a = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t)dt \tag{2.8.10}$$

where, $\boldsymbol{\lambda}(t)$ is the Lagrange multiplier (also called the costate function), and $\mathcal{L}$ is the Lagrangian given by

$$\boxed{\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t).} \tag{2.8.11}$$

Now, we just *use the results of the previous Stage I* for the augmented functional (2.8.10) except its integrand is $\mathcal{L}$ instead of V. For optimal condition, we have the Euler-Lagrange equation (2.8.3) for the augmented functional (2.8.10) given in terms of $\mathbf{x}(t)$ and $\boldsymbol{\lambda}(t)$ as

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0} \quad \textit{state} \text{ equation and} \tag{2.8.12}$$

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}}\right)_* = 0} \quad \textit{costate} \text{ equation.} \tag{2.8.13}$$

Let us note from (2.8.11) that the Lagrangian $\mathcal{L}$ is independent of $\dot{\boldsymbol{\lambda}}^*(t)$ and that the Euler-Lagrange equation (2.8.13) for the costate $\boldsymbol{\lambda}(t)$ is nothing but the constraint relation (2.8.9). The general boundary condition (2.8.4) to be satisfied at the free-final point (in terms of $\mathcal{L}$) is given by

$$\boxed{\left[\mathcal{L} - \dot{\mathbf{x}}'(t)\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)\right]_{*t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_{*t_f} \delta \mathbf{x}_f = 0.} \tag{2.8.14}$$

This boundary condition is to be modified depending on the nature of the given t_f and $\mathbf{x}(t_f)$.

2.8.3 Stage III: Optimal Control System with Lagrangian Formalism

Here, we consider the standard control system with a plant described by [56]

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{2.8.15}$$

the given boundary conditions as

$$\mathbf{x}(t_0) \text{ is fixed and } \mathbf{x}(t_f) \text{ is free,} \tag{2.8.16}$$

and the performance index as

$$J(\mathbf{u}(t)) = \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt. \tag{2.8.17}$$

Now, we rewrite the plant equation (2.8.15) as the condition relation (2.8.9) as

$$\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t) = 0. \tag{2.8.18}$$

Then we form the augmented functional out of the performance index (2.8.17) and the condition relation (2.8.18) as

$$J_a(\mathbf{u}(t)) = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t)dt \tag{2.8.19}$$

where, the Lagrangian $\mathcal{L}$ is given as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)\right\}. \end{aligned} \tag{2.8.20}$$

Now we *just use the previous results of Stage II.* For optimal condition, we have the set of Euler-Lagrange equations (2.8.12) and (2.8.13) for the augmented functional (2.8.19) given in terms of $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$, and $\mathbf{u}(t)$ as

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0} \quad \textit{state} \text{ equation,} \tag{2.8.21}$$

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}}\right)_* = 0} \quad \textit{costate} \text{ equation, and} \tag{2.8.22}$$

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{u}}}\right)_* = 0} \quad \textit{control} \text{ equation.} \tag{2.8.23}$$

Note from (2.8.20) that the Lagrangian $\mathcal{L}$ is independent of $\dot{\boldsymbol{\lambda}}^*(t)$ and $\dot{\mathbf{u}}^*(t)$ and that the Euler-Lagrange equation (2.8.22) is the same as the constraint relation (2.8.18). The general boundary condition (2.8.14) to be satisfied at the free-final point becomes

$$\boxed{\left[\mathcal{L} - \dot{\mathbf{x}}'(t)\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)\right]_{*t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_{*t_f} \delta \mathbf{x}_f = 0.} \tag{2.8.24}$$

This boundary condition is to be modified depending on the nature of the given t_f and $\mathbf{x}(t_f)$.

2.8.4 Stage IV: Optimal Control System with Hamiltonian Formalism: Pontryagin Principle

Here, we just transform the previous Lagrangian formalism to Hamiltonian formalism by defining the Hamiltonian as [57]

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.8.25}$$

which in terms of the Lagrangian (2.8.20) becomes

$$\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = \mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) - \boldsymbol{\lambda}'(t)\dot{\mathbf{x}}(t). \tag{2.8.26}$$

Now using (2.8.26), the set of Euler-Lagrange equations (2.8.21) to (2.8.23) which are in terms of the Lagrangian, are rewritten in terms of the Hamiltonian as

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}(-\boldsymbol{\lambda}^*) = 0 \qquad (2.8.27)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* - \dot{\mathbf{x}}^*(t) - \frac{d}{dt}(0) = 0 \qquad (2.8.28)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* - \frac{d}{dt}(0) = 0 \qquad (2.8.29)$$

which in turn are rewritten as

$$\boxed{\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*} \quad \textit{state} \text{ equation,} \qquad (2.8.30)$$

$$\boxed{\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*} \quad \textit{costate} \text{ equation, and} \qquad (2.8.31)$$

$$\boxed{0 = +\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*} \quad \textit{control} \text{ equation.} \qquad (2.8.32)$$

Similarly using (2.8.26), the boundary condition (2.8.24) is rewritten in terms of the Hamiltonian as

$$\left[\mathcal{H} - \boldsymbol{\lambda}'(t)\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}'(t)(-\boldsymbol{\lambda}(t))\right]\Big|_{*t_f} \delta t_f + \left[-\boldsymbol{\lambda}'(t)\right]\Big|_{*t_f} \delta \mathbf{x}_f = 0 \quad (2.8.33)$$

which becomes

$$\boxed{\mathcal{H}\big|_{*t_f} \delta t_f - \boldsymbol{\lambda}^{*\prime}(t_f)\delta \mathbf{x}_f = 0.} \qquad (2.8.34)$$

The sufficient condition is

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* > 0} \quad \text{for minimum and} \qquad (2.8.35)$$

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* < 0} \quad \text{for maximum.} \qquad (2.8.36)$$

The state, costate, and control equations (2.8.30) to (2.8.32) are solved along with the given initial condition (2.8.16) and the final condition (2.8.34) leading us to a two-point boundary value problem (TPBVP).

Free-Final Point System with Final Cost Function

This problem is an extension of the problem in Stage IV, with the addition of final cost function. We summarize the result risking the repetition of some of the equations. Let the plant be described as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.8.37}$$

and the performance index be

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.8.38}$$

along with the boundary conditions

$$\mathbf{x}(t_0) \text{ is fixed and } \mathbf{x}(t_f) \text{ is free.} \tag{2.8.39}$$

Now, if we rewrite the performance index (2.8.38) to absorb the final cost function S within the integrand, then the results of Stage III can be used to get the optimal conditions. Thus we rewrite (2.8.38) as

$$J_1(\mathbf{u}(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}(t), \mathbf{u}(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)' \dot{\mathbf{x}}(t) + \frac{\partial S}{\partial t} \right] dt. \tag{2.8.40}$$

Now we *repeat the results of Stage III* except for the modification of the final condition equation (2.8.34). Thus the state, costate and control equations are

$$\boxed{\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*} \quad \textit{state} \text{ equation} \tag{2.8.41}$$

$$\boxed{\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*} \quad \textit{costate} \text{ equation} \tag{2.8.42}$$

$$\boxed{0 = +\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*} \quad \textit{control} \text{ equation} \tag{2.8.43}$$

and the final boundary condition is

$$\boxed{\left[\mathcal{H} + \frac{\partial S}{\partial t}\right]_{*t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right) - \boldsymbol{\lambda}(t)\right]'_{*t_f} \delta \mathbf{x}_f = 0.} \tag{2.8.44}$$

The sufficient condition for optimum is

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* > 0} \quad \text{for minimum and} \tag{2.8.45}$$

$$\boxed{\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2}\right)_* < 0} \quad \text{for maximum.} \tag{2.8.46}$$

The state, costate, and control equations (2.8.41) to (2.8.43) are solved along with the given initial condition (2.8.39) and the final condition (2.8.44), thus this formulation leads us to a TPBVP.

2.8.5 Salient Features

We now discuss the various features of the methodology used so far in obtaining the optimal conditions through the use of the calculus of variations [6, 79, 120, 108]. Also, we need to consider the problems discussed above under the various stages of development. So we refer to the appropriate relations of, say Stage III or Stage IV during our discussion.

1. *Significance of Lagrange Multiplier:* The Lagrange multiplier $\boldsymbol{\lambda}(t)$ is also called the costate (or adjoint) function.

   (a) The Lagrange multiplier $\boldsymbol{\lambda}(t)$ is introduced to "take care of" the constraint relation imposed by the plant equation (2.8.15).

   (b) The costate variable $\boldsymbol{\lambda}(t)$ enables us to use the Euler-Lagrange equation for each of the variables $\mathbf{x}(t)$ and $\mathbf{u}(t)$ separately as if they were *independent* of each other although they are *dependent* of each other as per the plant equation.

2. *Lagrangian and Hamiltonian:* We defined the Lagrangian and Hamiltonian as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), \mathbf{u}(t), t) \\ &= V(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\quad + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)\right\}\end{aligned} \tag{2.8.47}$$

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\quad + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t).\end{aligned} \tag{2.8.48}$$

In defining the Lagrangian and Hamiltonian we use extensively the *vector* notation, still it should be noted that these $\mathcal{L}$ and $\mathcal{H}$ are *scalar* functions only.

3. *Optimization of Hamiltonian*

   (a) The control equation (2.8.32) indicates the optimization of the Hamiltonian w.r.t. the control $\mathbf{u}(t)$. That is, the optimization of the original performance index (2.8.17), which is a *functional* subject to the plant equation (2.8.15), is equivalent to the optimization of the Hamiltonian *function* w.r.t. $\mathbf{u}(t)$. Thus, we "reduced" our original *functional* optimization problem to an ordinary *function* optimization problem.

   (b) We note that we assumed *unconstrained or unbounded* control $\mathbf{u}(t)$ and obtained the control relation $\partial\mathcal{H}/\partial\mathbf{u} = 0$.

   (c) If $\mathbf{u}(t)$ is *constrained or bounded* as being a member of the set $\mathbf{U}$, i.e., $\mathbf{u}(t) \in \mathbf{U}$, we can no longer take $\partial\mathcal{H}/\partial\mathbf{u} = 0$, since $\mathbf{u}(t)$, so calculated, may lie outside the range of the permissible region $\mathbf{U}$. In practice, the control $\mathbf{u}(t)$ is always *limited* by such things as saturation of amplifiers, speed of a motor, or thrust of a rocket. The constrained optimal control systems are discussed in Chapter 7.

   (d) Regardless of any constraints on $\mathbf{u}(t)$, Pontryagin had shown that $\mathbf{u}(t)$ must still be chosen to minimize the Hamiltonian. A rigorous proof of the fact that $\mathbf{u}(t)$ must be chosen to optimize $\mathcal{H}$ function is Pontryagin's most notable contribution to optimal control theory. For this reason, the approach is often called *Pontryagin Principle*. So in the case of *constrained* control, it is shown that

$$\boxed{\min_{\mathbf{u} \in \mathbf{U}} \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}(t), t) = \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}^*(t), t)} \tag{2.8.49}$$

   or equivalently

$$\boxed{\mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}^*(t), t) \le \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), \mathbf{u}(t), t).} \tag{2.8.50}$$

4. *Pontryagin Maximum Principle:* Originally, Pontryagin used a slightly different performance index which is *maximized* rather

than *minimized* and hence it is called *Pontryagin Maximum Principle.* For this reason, the Hamiltonian is also sometimes called Pontryagin $\mathcal{H}$-function. Let us note that *minimization* of the performance index J is equivalent to the *maximization* of $-J$. Then, if we define the Hamiltonian as

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = -V(\mathbf{x}(t), \mathbf{u}(t), t) + \hat{\boldsymbol{\lambda}}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.8.51}$$

we have *Maximum Principle.* Further discussion on Pontryagin Principle is given in Chapter 6.

5. *Hamiltonian at the Optimal Condition:* At the optimal condition the Hamiltonian can be written as

$$\begin{aligned}\mathcal{H}^* &= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t)\\ \frac{d\mathcal{H}^*}{dt} &= \frac{d\mathcal{H}^*}{dt}\\ &= \left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*' \dot{\boldsymbol{\lambda}}^*(t) + \left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*' \dot{\mathbf{u}}^*(t)\\ &\quad + \left(\frac{\partial \mathcal{H}}{\partial t}\right)_*.\end{aligned} \tag{2.8.52}$$

Using the state, costate and control equations (2.8.30) to (2.8.32) in the previous equation, we get

$$\left(\frac{d\mathcal{H}}{dt}\right)_* = \left(\frac{\partial \mathcal{H}}{\partial t}\right)_*. \tag{2.8.53}$$

We observe that along the optimal trajectory, the *total* derivative of $\mathcal{H}$ w.r.t. time is the same as the *partial* derivative of $\mathcal{H}$ w.r.t. time. If $\mathcal{H}$ does not depend on t explicitly, then

$$\boxed{\left.\frac{d\mathcal{H}}{dt}\right|_* = 0} \tag{2.8.54}$$

and $\mathcal{H}$ is constant w.r.t. the time t along the optimal trajectory.

6. *Two-Point Boundary Value Problem (TPBVP):* As seen earlier, the optimal control problem of a dynamical system leads to a TPBVP.

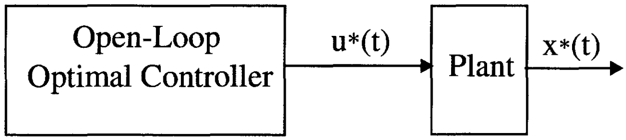

Figure 2.15 Open-Loop Optimal Control

(a) The state and costate equations (2.8.30) and (2.8.32) are solved using the initial and final conditions. In general, these are nonlinear, time varying and we may have to resort to numerical methods for their solutions.

(b) We note that the state and costate equations are the same for any kind of boundary conditions.

(c) For the optimal control system, although obtaining the state and costate equations is very easy, the computation of their solutions is quite tedious. This is the unfortunate feature of optimal control theory. It is the *price* one must pay for demanding the *best* performance from a system. One has to weigh the optimization of the system against the computational burden.

7. *Open-Loop Optimal Control:* In solving the TPBVP arising due to the state and costate equations, and then substituting in the control equation, we get only the open-loop optimal control as shown in Figure 2.15. Here, one has to construct or realize an open-loop optimal controller (OLOC) and in many cases it is very tedious. Also, changes in plant parameters are not taken into account by the OLOC. This prompts us to think in terms of a closed-loop optimal control (CLOC), i.e., to obtain optimal control $\mathbf{u}^*(t)$ in terms of the state $\mathbf{x}^*(t)$ as shown in Figure 2.16. This CLOC will have many advantages such as sensitive to plant parameter variations and simplified construction of the controller. The closed-loop optimal control systems are discussed in Chapter 7.

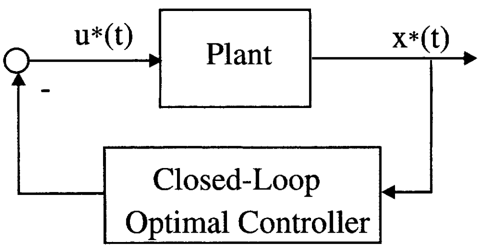

Figure 2.16 Closed-Loop Optimal Control

2.9 Problems

1. Make reasonable assumptions wherever necessary.
2. Use MATLAB© wherever possible to solve the problems and plot all the optimal controls and states for all problems. Provide the relevant MATLAB© *m* files.

Problem 2.1 Find the extremal of the following functional

$$J = \int_0^2 [2x^2(t) + \dot{x}^2(t)]dt$$

with the initial condition as $x(0) = 0$ and the final condition as $x(2) = 5$.

Problem 2.2 Find the extremal of the functional

$$J = \int_{-2}^{0} \left[12tx(t) + \dot{x}^2(t)\right] dt$$

to satisfy the boundary conditions $x(-2) = 3$, and $x(0) = 0$.

Problem 2.3 Find the extremal for the following functional

$$J = \int_1^2 \frac{\dot{x}^2(t)}{2t^3} dt$$

with $x(1) = 1$ and $x(2) = 10$.

Problem 2.4 Consider the extremization of a functional which is dependent on derivatives higher than the first derivative $\dot{x}(t)$ such as

$$J(x(t), t) = \int_{t_0}^{t_f} V(x(t), \dot{x}(t), \ddot{x}(t), t)dt.$$

with fixed-end point conditions. Show that the corresponding Euler-Lagrange equation is given by

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial V}{\partial \ddot{x}}\right) = 0.$$

Similarly, show that, in general, for extremization of

$$J = \int_{t_0}^{t_f} V\left(x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(r)}(t), t\right) dt$$

with fixed-end point conditions, the Euler-Lagrange equation becomes

$$\sum_{i=0}^{r} (-1)^i \frac{d^i}{dt^i}\left(\frac{\partial V}{\partial x^{(i)}}\right) = 0.$$

Problem 2.5 A first order system is given by

$$\dot{x}(t) = ax(t) + bu(t)$$

and the performance index is

$$J = \frac{1}{2}\int_0^{t_f} (qx^2(t) + ru^2(t))dt$$

where, $x(t_0) = x_0$ and $x(t_f)$ is free and t_f being fixed. Show that the optimal state $x^*(t)$ is given by

$$x^*(t) = x_0 \frac{sinh\beta(t_f - t)}{sinh\beta t_f}, \qquad \beta = \sqrt{a^2 + b^2 q/r}.$$

Problem 2.6 A mechanical system is described by

$$\ddot{x}(t) = u(t)$$

find the optimal control and the states by minimizing

$$J = \frac{1}{2}\int_0^5 u^2(t)dt$$

such that the boundary conditions are

$$x(t=0) = 2; \quad x(t=5) = 0; \quad \dot{x}(t=0) = 2; \quad \dot{x}(t=5) = 0.$$

Problem 2.7 For the first order system

$$\frac{dx}{dt} = -x(t) + u(t)$$

find the optimal control $u^*(t)$ to minimize

$$J = \int_0^{t_f} [x^2(t) + u^2(t)]dt$$

where, t_f is unspecified, and $x(0) = 5$ and $x(t_f) = 0$. Also find t_f.

Problem 2.8 Find the optimal control $u^*(t)$ of the plant

$$\begin{aligned}\dot{x}_1(t) &= x_2(t); \quad x_1(0) = 3, \quad x_1(2) = 0\\ \dot{x}_2(t) &= -2x_1(t) + 5u(t); \quad x_2(0) = 5, \quad x_2(2) = 0\end{aligned}$$

which minimizes the performance index

$$J = \frac{1}{2}\int_0^2 \left[x_1^2(t) + u^2(t)\right] dt.$$

Problem 2.9 A second order plant is described by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -2x_1(t) - 3x_2(t) + 5u(t)\end{aligned}$$

and the cost function is

$$J = \int_0^\infty [x_1^2(t) + u^2(t)]dt.$$

Find the optimal control, when $x_1(0) = 3$ and $x_2(0) = 2$.

Problem 2.10 For a second order system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -2x_1(t) + 3u(t)\end{aligned}$$

with performance index

$$J = 0.5x_1^2(\pi/2) + \int_0^{\pi/2} 0.5u^2(t)dt$$

and boundary conditions $\mathbf{x}(0) = [0 \;\; 1]'$ and $\mathbf{x}(t_f)$ is free, find the optimal control.

Problem 2.11 Find the optimal control for the plant

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -2x_1(t) + 3u(t)\end{aligned}$$

with performance criterion

$$\begin{aligned}J = &\frac{1}{2}F_{11}\left[x_1(t_f) - 4\right]^2 + \frac{1}{2}F_{22}\left[x_2(t_f) - 2\right]^2\\ &+\frac{1}{2}\int_0^{t_f}\left[x_1^2(t) + 2x_2^2(t) + 4u^2(t)\right] dt\end{aligned}$$

and initial conditions as $\mathbf{x}(0) = [1 \;\; 2]'$. The additional conditions are given below.

1. Fixed-final conditions $F_{11} = 0, F_{22} = 0, t_f = 2, \mathbf{x}(2) = [4 \;\; 6]'$.

2. Free-final time conditions $F_{11} = 3, F_{22} = 5, \mathbf{x}(t_f) = [4 \;\; 6]'$ and t_f is free.

3. Free-final state conditions, $F_{11} = F_{22} = 0, x_1(2)$ is free and $x_2(2) = 6$.

4. Free-final time and free-final state conditions, $F_{11} = 3, F_{22} = 5$ and the final state to have $x_1(t_f) = 4$ and $x_2(t_f)$ to lie on $\theta(t) = -5t + 15$.

Problem 2.12 For the D.C. motor speed control system described in Problem 1.1, find the open-loop optimal control to keep the speed constant at a particular value and the system to respond for any disturbances from the regulated value.

Problem 2.13 For the liquid-level control system described in Problem 1.2, find the open-loop optimal control to keep the liquid level constant at a reference value and the system to act only if there is a change in the liquid level.

Problem 2.14 For the inverted pendulum control system described in Problem 1.3, find the open-loop, optimal control to keep the pendulum in a vertical position.

Problem 2.15 For the mechanical control system described in Problem 1.4, find the open-loop, optimal control to keep the system at equilibrium condition and act only if there is a disturbance.

Problem 2.16 For the automobile suspension control system described in Problem 1.5, find the open-loop, optimal control to provide minimum control energy and passenger comfort.

Problem 2.17 For the chemical control system described in Problem 1.6, find the open-loop, optimal control to keep the system at equilibrium condition and act only if there is a disturbance.

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